# Control of Active Brownian Particles: An exact solution

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Control of stochastic systems is a challenging open problem in statistical physics, with potential applications in a wealth of systems from biology to granulates. Unlike most cases investigated so far, we aim here at controlling a genuinely out-of-equilibrium system, the two dimensional Active Brownian Particles model in a harmonic potential, a paradigm for the study of self-propelled bacteria. We search for protocols for the driving parameters (stiffness of the potential and activity of the particles) bringing the system from an initial passive-like stationary state to a final active-like one, within a chosen time interval. The exact analytical results found for this prototypical system of self-propelled particles brings control techniques to a wider class of out-of-equilibrium systems.

*Introduction* — Active matter is one of the most studied and promising topics of out-of-equilibrium statistical physics [1–4]. Inspired by the behaviour of biological systems such as bacteria and cells, this class of problems is characterized by internal mechanisms (e.g., self-propulsion) inducing nonzero entropy production, through energy dissipation. They are intrinsically out of equilibrium. Motility-induced phase separation [5, 6], pattern formation [7, 8], velocity self-alignment [9] are typical hallmarks of active models. More generally, self-propulsion and activity rule a wealth of systems, resulting in nano-swimming [10], complex colloidal or bacteria dynamics [11, 12], and active transport in biology [13]. While the engineering of such systems becomes possible [4], it remains a challenge to control activity in general; the present work is a step in this direction. This demands a proper understanding of the dynamics under confinement, an important endeavour for active objects [14, 15].

Several experiments have shown the possibility to tune the degree of activity of active matter [16–20]. In Ref. [16], for instance, silica spheres of a few $\mu m$ radius, partly covered by chromium and gold (Janus particles) are diluted in a binary mixture of water and 2,6-lutidine, that reacts with the surface of the particles and induces self-propulsion. The reaction is tuned by the intensity of light, so that the persistent velocity can be controlled. This light-dependent tuning is a promising mechanism for the control of active fluids and may have useful applications, e.g., for the clogging/unclogging of microchannels [21, 22]. The main idea behind these applications is to bring the system from a passive-like to an active-like phase, and vice-versa, and to take advantage of the different distribution of the particles in the two states.

The time needed to switch the system from one phase to the other will depend, in general, on the protocol that is employed to change the values of the controlling parameters. A sudden change of the external light, for instance, may then require a long time for the relaxation of the system to the desired final distribution. It is thus important to search for protocols that allow to execute the transition in a controlled way, in a short time. This type of problems, that can be subsumed under the terminology of "swift state-to state transformations" (SST) [23], has witnessed a surge of interest in the last 15 years. The first studies are in the realm of quantum mechanics [24], where they are referred to as "shortcuts to adiabaticity"; applications to statistical physics and stochastic thermodynamics are more recent [23, 25].

In this letter, we study such SST problems for a system of Active Brownian Particles (ABP) in two dimensions [14, 26–31]. This is one of the simplest and most used models mimicking the behaviour of self-propelled particles like bacteria [26], whose fluctuating hydrodynamics has been shown to be equivalent to the Run-and-Tumble model describing the above mentioned Janus particles [27, 32]. We will assume that the system is confined in an external harmonic potential with tunable stiffness, as done for instance in Ref. [33] by using acoustic waves. The stationary distribution of this model was found in Ref. [34]. With these assumptions, we will describe a class of analytical protocols leading the system from a passive-like to an active-like state with the same stiffness in a finite time, and vice-versa. Among this class of control protocols, we will identify the one minimizing the total time required for the transition.

*Model* — The state of 2 dimensional ABP is defined by a spatial position $\boldsymbol{\rho} = (\rho\cos\varphi, \rho\sin\varphi)$ for the center of mass, and an angle $\theta$ associated to the orientation of the particle. The particle's velocity is given by the sum of a self-propulsion contribution along the direction of $\theta$, $\mathbf{e}(\theta)$, with constant modulus $u_0$, plus a thermal Gaussian noise. The orientation $\theta$ is also subject to Gaussian fluctuations. In addition, the effect of an external potential will be taken into account. We consider the case of isotropic harmonic confinement, resulting in a force $-k\boldsymbol{\rho}$ pointing toward the origin ($k$ being the stiffness). In the overdamped limit, the time evolution is then given by the coupled Langevin equations

$$\begin{cases} \frac{d\boldsymbol{\rho}}{d\tau} = & u_0\widehat{\mathbf{e}}(\theta) - \mu k\boldsymbol{\rho} + \sqrt{2D_t}\,\boldsymbol{\xi}_r(\tau), \\ \frac{d\theta}{d\tau} = & \sqrt{2D_\theta}\,\boldsymbol{\xi}_\theta(\tau)\,, \end{cases} \tag{1}$$

where $\tau$ is the time, $\mu$ stands for the mobility, $\boldsymbol{\xi}_r(\tau)$ and $\boldsymbol{\xi}_\theta(\tau)$ are Gaussian white noises, while $D_t$ and $D_\theta$ are the translational and the rotational diffusivities. We also switch to the dimensionless variables $\mathbf{r} = \sqrt{D_\theta/D_t}\boldsymbol{\rho}$ and $t = D_\theta \tau$. As a matter of fact, it is convenient to recast the problem in terms of a Fokker-Planck equation for the evolution of the probability density function (pdf):

$$\partial_t P(\mathbf{r},\theta) = -\boldsymbol{\nabla}\left\{\left[\lambda\widehat{\mathbf{e}}(\theta) - \beta\mathbf{r} - \boldsymbol{\nabla}\right]P(\mathbf{r},\theta)\right\} + \partial_\theta^2 P(\mathbf{r},\theta)\,, \tag{2}$$

where $\lambda = u_0/\sqrt{D_\theta D_t}$ is a Péclet number which accounts for the degree of activity of the system, and $\beta = \mu k/D_\theta$ can be regarded as a dimensionless stiffness. Let us call $\varphi$ the angle between the $x$ axis of the plane where the particle moves and the vector $\mathbf{r}$. Assuming isotropy, the pdf is expected to depend only on the difference $\chi = \theta - \varphi$. Eq. (2) can be thus recast as

$$\partial_t P = \mathcal{L}_r P - \lambda\cos\chi\,\partial_r P + \lambda\sin\chi\frac{1}{r}\partial_\chi P + \frac{1}{r^2}\partial_\chi^2 P + \partial_\chi^2 P \tag{3}$$

where

$$\mathcal{L}_r f(r) = \frac{1}{r}\partial_r\left[r\left(\partial_r + \beta r\right)f(r)\right]\,. \tag{4}$$

An analytical stationary solution can be worked out as a series expansion in powers of $\lambda$ [34]:

$$P_s(r,\chi|\beta,\lambda) = \sum_{m=0}^{\infty}\lambda^m \sum_{2n+|l|=m} C_{n,l}^{(m)}(\beta)\phi_{n,l}(r,\chi|\beta)\,. \tag{5}$$

Here, the $C_{n,|l|}^{(m)}(\beta)$ are coefficients that can be determined by suitable recursive rules, see Supplemental Material (SM) [35]. They are by construction independent of the activity parameter $\lambda$. The functions $\phi_{n,l}(r,\chi)$ are defined by

$$\phi_{n,l}(r,\chi|\beta) = \left[\frac{n!\left(\frac{\beta}{2}\right)^{|l|+1}}{\pi(n+|l|)!}\right]^{\frac{1}{2}} r^{|l|}e^{-\frac{\beta r^2}{2}}L_n^{|l|}\left(\frac{\beta r^2}{2}\right)e^{il\chi} \tag{6}$$

where $L_n^\alpha(x)$ are the generalized Laguerre polynomials. The second sum in Eq. (5) is constrained to the (integers) values of $l$ and $n \geq 0$ such that $2n + |l| = m$.

There are three relevant time scales for this model: $\tau_r = D_\theta^{-1}$ is the typical time for the rotation of a particle, $\tau_k = (\mu k)^{-1} = \tau_r/\beta_0$ is the relaxation time of the over-damped harmonic oscillator, and $\tau_u = \frac{1}{u_0}\sqrt{\frac{D_t}{D_\theta}} = \tau_r/\lambda$ is a typical time-scale of the activity [36], namely the time needed by a particle ballistically moving with velocity $u_0$ to cover the same distance as a passive particle would reach by diffusion during a "rotation" (i.e. in a time $\tau_r$). In typical experimental situations, where $\lambda > 1$ and $\beta > 1$ (see discussion below), $\tau_r$ is the largest. Spontaneous thermalization is expected to occur for $\tau \gg \tau_r$ [29] or, in our rescaled variables, $t \gg 1$.

*Swift state-to-state transformations* — We now face the problem of bringing the system from an initial stationary state, characterized by $\lambda = \lambda_i$, to a final state $\lambda = \lambda_f$ with the same $\beta = \beta_0$, in a given time interval $t_f$. We assume that the controlling parameters $\lambda$ and $\beta$ can be varied during the evolution. We impose, searching for an exact solution, that the form of the pdf during the process is described by the functional form:

$$P(r,\chi,t) \equiv \alpha(t)P_s\left(\sqrt{\alpha(t)}r,\chi|\beta_0,\sqrt{\alpha(t)}\lambda(t)\right)\,. \tag{7}$$

Here $\alpha(t)$ is an arbitrary continuous function of time such that $\alpha(t) > 0$ and

$$\alpha(0) = \alpha(t_f) = 1\,. \tag{8}$$

When $\alpha = 1$, the pdf corresponds to the stationary state induced by the external parameters $\lambda(t)$ and $\beta_0$. The normalization condition is verified in the SM [35]. We plug Eq. (7) into the Fokker-Planck equation (3). The calculations, carried out in the SM [35], are quite lengthy: they yield the conditions that the dimensionless activity $\lambda$ and stiffness $\beta$ should fulfill so that Eq. (7) be a valid solution. We find

$$\beta(t) = \frac{\dot\alpha(t)}{2\alpha(t)} + \beta_0\alpha(t) \tag{9a}$$

$$\dot\lambda(t) = -\left[\beta(t) - \beta_0\right]\lambda(t)\,, \tag{9b}$$

Once supplemented with the boundary conditions (8) and

$$\lambda(0) = \lambda_i \quad \lambda(t_f) = \lambda_f\,, \tag{10}$$

Eqs. (9) provide a class of eligible SST for the process. They constitute the main result of this letter. We remark that the substantial freedom for the choice of $\alpha(t)$ leads to a wide family of exact protocols for the controlled dynamics.

From a physical point of view, taking into account the definition of $\lambda$ and $\mathbf{r}$, ansatz (7) is equivalent to a stationary state where $\beta = \beta_0$ and the thermal diffusivity $D_t$ (or, the bath temperature) is divided by a factor $\alpha(t)$. In other words, we assume that during the dynamics, the instantaneous pdf has the same shape of a fictitious stationary state with fixed stiffness and varying $D_t$ and $u_0$: at the end of the process, $D_t$ is brought back to its original value so to match the actual bath temperature, while $u_0$ assumes a different value with respect to the initial one, allowing for a passive-to-active transition, or vice-versa.

*Controlled protocols* — As recalled in the Introduction, the degree of activity and the stiffness of the external potential can be controlled in actual experiments. With the setup studied by Buttinoni et al. [16], spherical Janus particles with radius $R = 1\,\mu m$ can have a persistent velocity varying in the interval $0\,\mu m/s \leq u_0 \leq 1\,\mu m/s$, depending on the intensity of the surrounding light. The rotational diffusivity has been measured to

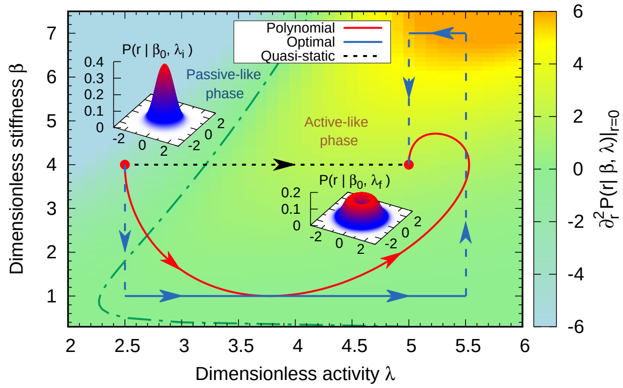


FIG. 1. Parameter space of the model. The color code represents the value of $\partial_r^2 P_s(r)|_{r=0}$, which is zero at the interface between the passive-like and the active-like phase (green dash-dotted curve). The black dotted line represents the path of a quasi-static protocol in which $\lambda$ is slowly varied between $\lambda_i = 2.5$ and $\lambda_f = 5$, while $\beta = \beta_0 = 4$ is kept constant. The red solid curve describes the solution to Eqs. (9) associated to the polynomial protocol (14); in this case the final time $t_f$ is chosen in such a way that $\beta(t)$ does not exceed the bounds $1 < \beta < 7$, inspired by the experimental constraints discussed in the text. The blue lines show the minimal-time protocol, the dashed branches representing instantaneous change in the control parameter $\beta$. Plots of the position pdf for the initial and the final states are also shown.

be $D_\theta \simeq 0.08\, s^{-1}$. Calling $\eta$ the dynamic viscosity of the fluid, $T$ the temperature and $k_B$ the Boltzmann constant, it is also possible to estimate the translational diffusivity of the particles (not measured in the paper):

$$D_t = \frac{k_B T}{6\pi\eta R} = \frac{4}{3} R^2 D_\theta \simeq 0.10\, \mu m^2/s\,, \tag{11}$$

in agreement with the estimation provided in Ref. [33] for a similar situation. The dimensionless parameter $\lambda = u_0/\sqrt{D_\theta D_t}$ can be thus tuned in the interval

$$0 \le \lambda \le 11\,. \tag{12}$$

The particles may be confined in a quasi-harmonic, controllable potential as done in Ref. [33], where acoustic waves are employed to trap a system of Janus particles with different chemical properties but similar radius. In that paper, two experimental situations are studied, in which particles with $\tau_r$ between $2s$ and $5s$ attain states with $\beta = 0.29$ and $\beta = 1.76$. Taking into account the different characteristic time for rotations, the dimensionless stiffness for the system described in Ref. [16] can be expected to be tunable, at least, within the interval $1.2 \le \beta \le 7$. A lower bound to the stiffness is expected to hold in experimental setups to prevent particles from moving away from the trap.

In Fig. 1 the parameter space of the model is sketched. As in Ref. [34], we distinguish between a passive-like phase characterized by $\partial_r^2 P_s(0,\chi) < 0$ and an active-like one where the particles tend to escape from the center of the potential and $\partial_r^2 P_s(0,\chi) > 0$. The range of the control parameters that is expected to be reached in experiments includes both passive-like and active-like stationary distributions, and it is interesting to search for SST between these two states.

Possibly, the simplest way to find an explicit smooth protocol satisfying Eqs. (9) is to enforce a polynomial form for $\alpha(t)$. We have to impose the boundary conditions Eq. (8) and the final condition for $\lambda$. If we also require that

$$\beta(0) = \beta(t_f) = \beta_0\,, \tag{13}$$

i.e. that the stiffness is varied continuously without abrupt changes at the beginning and at the end of the protocol, five degrees of freedom are needed. The polynomial needs therefore to be at least fourth order, $\alpha(t) = \sum_{n=0}^{4} \alpha_n t^n$, and one has

$$\begin{gathered}\alpha_0 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{30 \ln[\lambda(t_f)/\lambda(0)]}{\beta_0 t_f^3}, \\ \alpha_3 = -\frac{2\alpha_2}{t_f}, \quad \alpha_4 = \frac{\alpha_2}{t_f^2}\,.\end{gathered} \tag{14}$$

In Fig. 1, the red solid curve shows a protocol of this sort for a realistic situation, bringing the state of the system from the passive- to the active-like phase in a time $t_f \simeq 0.66$, where an abrupt change of the parameters would have led to a thermalization on time-scales $t \gg 1$, as discussed before.

*Minimal time* — Our interest now goes to finding the optimum protocol, i.e. with the shortest connecting time $t_f^{min}$, among all those encoded in the form (7). This amounts to identifying the optimal function $\alpha(t)$, from which the driving parameters (stiffness $\beta(t)$ and activity $\lambda(t)$) follow. Integrating Eq. (9b), one has

$$\ln\frac{\lambda_f}{\lambda_0} = \beta_0 \int_0^{t_f} dt' [1 - \alpha(t')]\,, \tag{15}$$

i.e. the area between $\alpha(t)$ and the line $\alpha = 1$ is determined once $\lambda_0$, $\lambda_f$ and $\beta_0$ are fixed, and it does not depend on $t_f$. Assuming that $\beta$ is bounded by $\beta_- \le \beta \le \beta_+$, this condition allows to find the minimal time for the protocol. We will consider the case in which the activity of the particles is increased during the process (the inverse process can be derived similarly). The constraints on $\beta$, recalling Eq. (9a), lead to

$$\beta_- \le \frac{\dot\alpha(t)}{2\alpha(t)} + \beta_0\alpha(t) \le \beta_+\,. \tag{16}$$

The two limiting curves $\alpha_-(t)$ and $\alpha_+(t)$ (obtained by setting $\beta(t) = \beta_-$ and $\beta(t) = \beta_+$ respectively) are:

$$\alpha_-(t) = \frac{\beta_-}{\beta_0 - (\beta_0 - \beta_-)e^{-2\beta_- t}} \tag{17}$$

$$\alpha_+(t) = \frac{\beta_+}{\beta_0 - (\beta_0 - \beta_+)e^{2\beta_+(t_f^{min} - t)}}\,, \tag{18}$$

where the boundary conditions Eq. (8) have been enforced. In this case we assume that the stiffness of the confining potential can be varied on time scales much shorter than the typical times of the dynamics, so that we do not have to take into account the boundary conditions (13).

Since $\alpha_-$ ($\alpha_+$) is monotonically decreasing (increasing), the best strategy to cover the area prescribed by Eq. (15) in the minimum time is to alternate a maximal decompression ($\alpha(t) = \alpha_-(t)$) and a maximal compression ($\alpha(t) = \alpha_+(t)$). This approach is usually referred to as "bang-bang protocol" [37]. Let us denote by $t^\star$ the time at which the two regimes are switched. The continuity condition on $\alpha(t)$ yields

$$\alpha_-(t^\star) = \alpha_+(t^\star) \equiv \alpha^\star\,, \tag{19}$$

while from Eq. (15) one obtains, by integration,

$$\ln\left(\frac{\lambda_f}{\alpha^\star \lambda_i}\right) = \beta_0 t_f^{min} - \beta_- t^\star - \beta_+(t_f^{min} - t^\star)\,. \tag{20}$$

The above equations can be solved numerically for $t^\star$ and $t_f^{min}$ (see SM [35] for a plot of $t_f^{min}$ as a function of the boundary conditions). In figure 1 the blue curve represents the optimal protocol in the parameter space under some realistic constraints. The time dependence of the parameters is presented in Fig. 2, where also the smooth polynomial protocol discussed before is shown for comparison. In panel 2(c) the equivalence of the areas discussed above can be appreciated for the two considered processes.

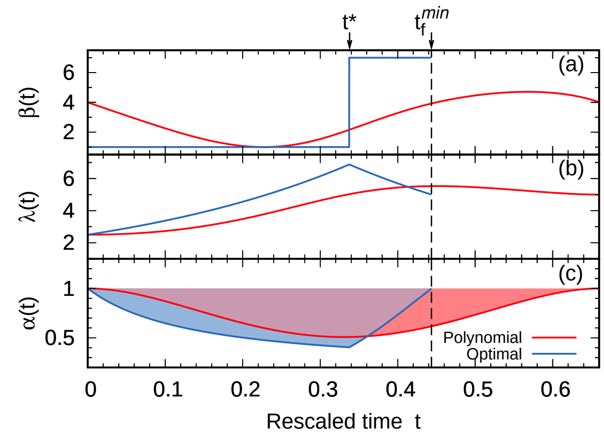


FIG. 2. Evolution of the parameters for the minimum-time protocol and the the solution to Eqs. (9) associated to the polynomial with coefficients (14). Panel (a) shows the time dependence of $\beta(t)$, panel (b) that of $\lambda(t)$ and panel (c) the evolution of $\alpha(t)$. The dashed vertical line identifies the minimum time over all possible protocols of the type given by Eq. (7), with $1 < \beta < 7$. The switching time $t^\star$ is also highlighted on the top axis. In both cases $t_f < 1$, while the relaxation following an abrupt change of the controlling parameters would take a time $t_f \gg 1$. The shaded areas in panel (c) do not depend on the protocol, once $\beta_0$ and $\lambda_f/\lambda_i$ are fixed, as prescribed by Eq. (15).

*Conclusions* — In the present letter, we have discussed a class of exact analytical protocols to bring an ABP system from an initial non-equilibrium stationary state to another final stationary state having a different degree of activity, in a given time. Among this family of protocols, we have also identified the one leading to the minimal time. The proposed protocols are expected to be applicable in actual experiments with tunable active particles, with possible applications in the context of clogging [22]: externally tuning the shape of the particle distribution induces control of the flux of fluid or the quantity of light passing through the system. We emphasize that our method, based on suitable deformations of the stationary distribution, may be used to search for SST ("swift state-to state transformations") in different contexts, provided that the stationary state is known. Finding the general conditions to be fulfilled for this approach to provide a suitable solution is an interesting research perspectives.

Our computation is the starting point for the solution of other optimal problems for ABPs: for instance, the average work done during a realization can be computed [35] and minimized with analytical methods, a task that has been so far accomplished, for active models, only with numerical techniques [38]. Since our search for the optimal protocol is restricted to the class of processes fulfilling condition (7), a further step would consist in proving (or excluding) that the "global" optimum belongs to this family, making use of Pontryagin's principle [39]. Protocols connecting states with different stiffness may be also searched for, following similar approaches. Future developments pertain to the search for SSTs in three dimensions [40] (e.g., in the presence of homogeneous external force [41]), and for interacting particles [14, 28]; the latter has been studied in the context of passive systems [42], but with few degrees of freedom only.

To summarize, we have extended the quest for controlling stochastic motion to the realm of active particles. To the best of our knowledge, this is the first case in which SST can be found for this class of systems, and one of the few involving out-of-equilibrium models [37, 43, 44]. Similar strategies may be attempted for active particles models whose stationary state is analytically known, as the 1D Run-and-Tumble [45, 46] or the Active Ornstein-Uhlenbeck particles with Unified Color Noise approximation [47, 48]. As the results contained in the present work show, analytical techniques from the domain of control theory can be successfully applied to simple, but non-trivial, models of active matter, opening the way to a number of research directions in this field.

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# Supplemental Material for "Control of Active Brownian Particles: an exact solution"

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In Sections I and II we recall, for the sake of completeness, the definition of the coefficients $C_{n,l}^{(m)}(\beta)$ introduced in Ref. [1] and the properties of the Generalized Laguerre Polynomial, which will turn out to be useful for the following calculations. In Section III, we show that the ansatz made in the main text is properly normalized. In Section IV Eq. (9), the main result of the paper, is derived. Section V shows a plot of the minimal time as a function of the problem's parameters. Finally, Section VI is devoted to the derivation of the average work.

## I. DEFINITION OF THE COEFFICIENTS $C_{n,l}^{(m)}(\beta)$

As discussed in Ref. [1], the coefficients $C_{n,l}^{(m)}(\beta)$ can be determined recursively, according to the following iterative rules (the $\beta$ dependence is dropped to avoid clutter):

$$C_{0,0}^{(0)} = \sqrt{\frac{\beta}{2\pi}} \tag{1a}$$

$$C_{0,l}^{(l)} = \frac{C_{0,l-1}^{(l-1)}\sqrt{l\frac{\beta}{2}}}{\beta l + l^2} \qquad l > 0 \tag{1b}$$

$$C_{n,0}^{(2n)} = -\frac{C_{n-1,1}^{(2n-1)}}{\sqrt{2\beta n}} \qquad n > 0 \tag{1c}$$

$$C_{n,l}^{(m)} = \frac{C_{n,l-1}^{(m-1)}\sqrt{(n+|l|)\frac{\beta}{2}} - C_{n-1,|l|+1}^{(m-1)}\sqrt{n\frac{\beta}{2}}}{\beta(2n+|l|)+l^2} \qquad l > 0, n > 0\,. \tag{1d}$$

The coefficients with negative values of $l$ are defined by the symmetry relation

$$C_{n,-l}^{(m)} = C_{n,l}^{(m)} \qquad l > 0\,. \tag{2}$$

With the above prescriptions, one can first compute $C_{0,l}^{(l)}$ iteratively for every value of $l > 0$, then all terms $C_{1,l}^{l+1}$, $C_{2,l}^{l+2}$, and so on, and finally use the symmetry relation (2) to determine the remaining coefficients.

## II. SOME PROPERTIES OF THE GENERALIZED LAGUERRE POLYNOMIALS

The Generalized Laguerre polynomials are defined as

$$L_n^{(y)}(x) = \sum_{i=0}^{n} (-1)^i \binom{n+y}{n-i} \frac{x^i}{i!}\,, \tag{3}$$

or, alternatively, through the Rodrigues formula

$$L_n^{(y)}(x) = \frac{x^{-y}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+y}\right)\,. \tag{4}$$

These functions are a generalization of the Laguerre polynomials (which can be recovered by setting $y = 0$), hence the name.

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It can be shown that, with the above definition, the derivative of a generalized Laguerre polynomial assumes the form

$$\frac{d^k}{dx^k} L_n^{(y)}(x) = \begin{cases} (-1)^k L_{n-k}^{(y+k)}(x) & \text{if} \quad k \leq n \\ 0 & \text{otherwise} \,. \end{cases} \tag{5}$$

Several recurrence formulas can be defined for this set of functions. Two particularly useful ones are the following:

$$L_n^{(y)}(x) = L_n^{(y+1)}(x) - L_{n-1}^{(y+1)}(x) \tag{6}$$

$$L_n^{(y)}(x) = \frac{y+1-x}{n} L_{n-1}^{(y+1)}(x) - \frac{x}{n} L_{n-2}^{(y+2)}(x) \,. \tag{7}$$

## III. NORMALIZATION OF THE ANSATZ

Here we verify that the proposed ansatz is properly normalized for any choice of $\alpha$ and $\lambda$, i.e., we check that the quantity

$$\int_0^{2\pi} d\chi \int_0^{\infty} dr\, r P(r, \chi|\alpha, \lambda, \beta_0) = \int_0^{2\pi} d\chi \int_0^{\infty} dr\, r \sum_{m=0}^{\infty} \lambda^m \alpha^{\frac{m}{2}+1} \sum_{n,l}^{2n+|l|=m} C_{n,l}^{(m)}(\beta_0) \phi_{n,l}(\sqrt{\alpha} r, \chi|\beta_0) \tag{8}$$

with

$$\phi_{n,l}(r, \chi|\beta) = \left[ \frac{n! \left(\frac{\beta}{2}\right)^{|l|+1}}{\pi(n+|l|)!} \right]^{\frac{1}{2}} r^{|l|} e^{-\frac{\alpha\beta r^2}{2}} L_n^{|l|}\left(\frac{\beta r^2}{2}\right) e^{il\chi} \,, \tag{9}$$

is equal to 1, where the coefficients $C_{n,l}^{(m)}$ are defined in Sec. I of the present Supplemental Material (SM).

First of all, let us notice that all terms with $l \neq 0$ are null, due to the factor $e^{il\chi}$ which turns into a Kronecker delta $2\pi\delta_{l,0}$ once it is integrated over $\chi$. The second sum is constrained to $2n + |l| = m$, which implies therefore $m = 2n$ for all nonvanishing terms. We can thus write

$$\begin{aligned} \int_0^{2\pi} d\chi \int_0^{\infty} dr\, r P(r, \chi|\alpha, \lambda, \beta_0) =& \sqrt{2\pi\beta_0} \int_0^{\infty} dr\, r \sum_{n=0}^{\infty} \lambda^{2n} \alpha^{n+1} C_{n,0}^{(2n)}(\beta_0) e^{-\frac{\alpha\beta r^2}{2}} L_n^0\left(\frac{\alpha\beta_0 r^2}{2}\right) \\ =& \sqrt{\frac{2\pi}{\beta_0}} \sum_{n=0}^{\infty} \lambda^{2n} \alpha^n C_{n,0}^{(2n)}(\beta_0) \int_0^{\infty} du\, e^{-u} L_n^0(u) \\ =& \sqrt{\frac{2\pi}{\beta_0}} \sum_{n=0}^{\infty} \lambda^{2n} \alpha^n C_{n,0}^{(2n)}(\beta_0) \frac{1}{n!} \int_0^{\infty} du\, \frac{d^n}{du^n}\left(e^{-u} u^n\right) \,, \end{aligned} \tag{10}$$

where we first implemented the variable change $u = \alpha\beta_0 r^2/2$ and then, in the last step, we exploited the Rodrigues formula for Laguerre polynomials (see Eq. (4) of Appendix II). The integral is only different from zero (and equal to 1) when $n = 0$, so one finally has:

$$\int_0^{2\pi} d\chi \int_0^{\infty} dr\, r P(r, \chi|\alpha, \lambda, \beta_0) = \sqrt{\frac{2\pi}{\beta_0}} C_{0,0}^{(0)}(\beta_0) = 1 \,. \tag{11}$$

## IV. DERIVATION OF EQ. (9)

To derive Eq. (9) of the main text, one has to plug the ansatz

$$P(r, \chi|\alpha, \lambda, \beta_0) = \sum_{m=0}^{\infty} \lambda^m \alpha^{\frac{m}{2}+1} \sum_{n,l}^{2n+|l|=m} C_{n,l}^{(m)}(\beta_0) \phi_{n,l}(\sqrt{\alpha} r, \chi|\beta_0) \,, \tag{12}$$

with $\phi_{n,l}$ defined by Eq. (9), into the Fokker-Planck equation

$$\partial_t P = \mathcal{L}_r P - \lambda \cos\chi \partial_r P + \lambda \sin\chi \frac{1}{r}\partial_\chi P + \frac{1}{r^2}\partial_\chi^2 P + \partial_\chi^2 P\,. \tag{13}$$

To avoid clutter, in the following we will drop explicit dependence on the parameters, i.e in this SM we will always consider

$$P \equiv P(r,\chi|\alpha(t),\lambda(t),\beta_0) \quad C_{n,l}^{(m)} \equiv C_{n,l}^{(m)}(\beta_0) \quad \phi_{n,l} \equiv \phi_{n,l}(\sqrt{\alpha(t)}r,\chi|\beta_0) \quad L_n^l \equiv L_n^l\left(\frac{\alpha(t)\beta_0 r^2}{2}\right)\,.$$

We start with the time derivative $\partial_t P$. Our ansatz only depends on time through $\lambda(t)$ and $\alpha(t)$. One obtains

$$\partial_t P = \sum_{m=0}^{\infty} \lambda^m \sqrt{\alpha^{m+2}} \sum_{2n+|l|=m} C_{n,l}^{(m)} \left\{ m\frac{\dot\lambda}{\lambda} + \frac{\dot\alpha}{2\alpha}\left[|l| + m + 2 - \alpha\beta r^2 - \alpha\beta r^2 \frac{L_{n-1}^{|l|+1}}{L_n^{|l|}}\right]\right\}\phi_{n,l}\,, \tag{14}$$

where the rule for the derivative of generalized Laguerre polynomials, Eq. (5), has been exploited.

The term $\mathcal{L}_r P$ is more involved, and its evaluation requires the use of the recursion formulas for the generalized Laguerre Polynomials. It is important to note that inside the operator $\mathcal{L}_r$ one has the time-dependent parameter $\beta$ (which determines the stiffness of the external potential).

$$\begin{aligned}
\mathcal{L}_r P &= \sum_{m=0}^{\infty} \lambda^m \sqrt{\alpha^{m+2}} \sum_{2n+|l|=m} C_{n,l}^{(m)} \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial \phi_{n,l}}{\partial r} + \beta r \phi_{n,l}\right)\right] \\
&= \sum_{m=0}^{\infty} \lambda^m \sqrt{\alpha^{m+2}} \sum_{2n+|l|=m} C_{n,l}^{(m)} \frac{1}{r}\frac{\partial}{\partial r}\left[\left(|l| - \alpha\beta_0 r^2 \frac{L_{n-1}^{|l|+1}}{L_n^{|l|}} + (\beta-\alpha\beta_0) r\right)\phi_{n,l}\right] \\
&= \sum_{m=0}^{\infty} \lambda^m \sqrt{\alpha^{m+2}} \sum_{2n+|l|=m} C_{n,l}^{(m)} \left(\frac{l^2}{r^2} - \alpha\beta_0|l| + \left[-2(|l|+2)\alpha\beta_0 + \alpha^2\beta_0^2 r^2\right]\frac{L_{n-1}^{|l|+1}}{L_n^{|l|}} + \alpha^2\beta_0^2 r^2 \frac{L_{n-2}^{|l|+1}}{L_n^{|l|}}\right)\phi_{n,l} + \\
&\qquad + \sum_{m=0}^{\infty} \lambda^m \sqrt{\alpha^{m+2}} \sum_{2n+|l|=m} C_{n,l}^{(m)} (\beta-\alpha\beta_0)\left(|l| + 2 - \alpha\beta_0 r^2 - \alpha\beta_0 r^2 \frac{L_{n-1}^{|l|+1}}{L_n^{|l|}}\right)\phi_{n,l} \\
&= \sum_{m=0}^{\infty} \lambda^m \sqrt{\alpha^{m+2}} \sum_{2n+|l|=m} C_{n,l}^{(m)} \left[\frac{l^2}{r^2} - \alpha\beta_0|l| - 2n\alpha\beta_0 + (\beta-\alpha\beta_0)\left(|l| + 2 - \alpha\beta_0 r^2 - \alpha\beta_0 r^2\frac{L_{n-1}^{|l|+1}}{L_n^{|l|}}\right)\right]\phi_{n,l}
\end{aligned} \tag{15}$$

where, in the last step, we made use of Eq. (7).

Since the pdf $P$ only depends on $\chi$ through the factor $e^{il\chi}$ appearing in the definition of $\phi$, the terms of the Fokker-Planck equation which only involve the second derivative with respect to $\chi$ read

$$\frac{1}{r^2}\frac{\partial^2 P}{\partial \chi^2} + \frac{\partial^2 P}{\partial \chi^2} = \sum_{m=0}^{\infty} \lambda^m \sqrt{\alpha^{m+2}} \sum_{2n+|l|=m} C_{n,l}^{(m)} \left[-\frac{l^2}{r^2} - l^2\right]\phi_{n,l}\,. \tag{16}$$

Finally, we have to evaluate the $\lambda$-dependent terms. First, we have that

$$-\lambda\cos\chi\partial_r P = -\cos\chi \sum_{m=0}^{\infty} \lambda^{m+1}\sqrt{\alpha^{m+2}} \sum_{2n+|l|=m} C_{n,l}^{(m)} \left(|l| r^{-1} - \alpha\beta_0 r - \alpha\beta_0 r \frac{L_{n-1}^{|l|+1}}{L_n^{|l|}}\right)\phi_{n,l}\,. \tag{17}$$

Here we have to apply in sequence Eqs. (6) and (7), leading to

$$
\begin{aligned}
&-\lambda\cos\chi\partial_r P\\
&= -\cos\chi\sum_{m=0}^{\infty}\lambda^{m+1}\sqrt{\alpha^{m+2}}\sum_{2n+|l|=m}C_{n,l}^{(m)}\left((n+1)r^{-1}\frac{L_{n+1}^{|l|-1}}{L_n^{|l|}}-\frac{\alpha\beta_0 r}{2}\frac{L_n^{|l|+1}}{L_n^{|l|}}\right)\phi_{n,l}\\
&= -\sum_{n=0}^{\infty}\sum_{|l|>0}\delta_{m,2n+|l|}\lambda^{m+1}\sqrt{\alpha^{m+2}}C_{n,l}^{(m)}\cos\chi\left((n+1)r^{-1}\frac{L_{n+1}^{|l|-1}}{L_n^{|l|}}-\frac{\alpha\beta_0 r}{2}\frac{L_n^{|l|+1}}{L_n^{|l|}}\right)\phi_{n,l}\\
&\quad -\sum_{n=0}^{\infty}\lambda^{2n+1}\alpha^{n+1}C_{n,l}^{(2n)}\cos\chi\sqrt{\frac{\beta_0}{2\pi}}\left((n+1)r^{-1}L_{n+1}^{-1}-\frac{\alpha\beta_0 r}{2}L_n^1\right)e^{-\frac{\alpha\beta_0 r^2}{2}}\\
&= -\sum_{n=0}^{\infty}\sum_{|l|>0}\delta_{m,2n+|l|}\lambda^{m+1}\sqrt{\alpha^{m+2}}C_{n,l}^{(m)}\cos\chi\left((n+1)r^{-1}\frac{L_{n+1}^{|l|-1}}{L_n^{|l|}}-\frac{\alpha\beta_0 r}{2}\frac{L_n^{|l|+1}}{L_n^{|l|}}\right)\phi_{n,l}\\
&\quad +2\sum_{n=0}^{\infty}\lambda^{2n+1}\alpha^{n+1}C_{n,l}^{(2n)}\left(\frac{e^{i\chi}}{2}+\frac{e^{-i\chi}}{2}\right)\sqrt{\frac{\beta_0}{2\pi}}\frac{\alpha\beta_0 r}{2}L_n^1 e^{-\frac{\alpha\beta_0 r^2}{2}}\,.
\end{aligned}
\tag{18}
$$

In the last step, we made use of the relation

$$
(n+1)r^{-1}L_{n+1}^{-1}=-\frac{\alpha\beta_0 r}{2}L_n^0-\frac{\alpha\beta_0 r}{2}L_{n-1}^1=-\frac{\alpha\beta_0 r}{2}L_n^1
\tag{19}
$$

following from Eqs. (6) and (7).

Next, we have to evaluate

$$
\begin{aligned}
\lambda\frac{\sin\chi}{r}\partial_\chi P &= \sum_{m=0}^{\infty}\lambda^{m+1}\sqrt{\alpha^{m+2}}\sum_{2n+|l|=m}C_{n,l}^{(m)}\sin\chi\, il\phi_{n,l}\\
&= \sum_{n=0}^{\infty}\sum_{|l|>0}\delta_{m,2n+|l|}\lambda^{m+1}\sqrt{\alpha^{m+2}}C_{n,l}^{(m)}\frac{il}{|l|}\sin\chi\left((n+1)r^{-1}\frac{L_{n+1}^{|l|-1}}{L_n^{|l|}}+\frac{\alpha\beta_0 r}{2}+\frac{\alpha\beta_0 r}{2}\frac{L_{n-1}^{|l|+1}}{L_n^{|l|}}\right)\phi_{n,l}\\
&= \sum_{n=0}^{\infty}\sum_{|l|>0}\delta_{m,2n+|l|}\lambda^{m+1}\sqrt{\alpha^{m+2}}C_{n,l}^{(m)}\frac{il}{|l|}\sin\chi\left((n+1)r^{-1}\frac{L_{n+1}^{|l|-1}}{L_n^{|l|}}-\frac{\alpha\beta_0 r}{2}\frac{L_n^{|l|+1}}{L_n^{|l|}}\right)\phi_{n,l}\,,
\end{aligned}
\tag{20}
$$

where we first applied the recurrence formula (7) and then Eq. (6). Putting together Eq. (18) and (20) and noticing that

$$
\cos\chi\pm\frac{il}{|l|}\sin\chi=e^{\pm i\frac{l}{|l|}\chi}
\tag{21}
$$

one gets

$$
\begin{aligned}
&-\lambda\cos\chi\partial_r+\lambda\frac{\sin\chi}{r}\partial_\chi P=\\
&= -\sum_{n=0}^{\infty}\sum_{|l|>0}\delta_{m,2n+|l|}\lambda^{m+1}\sqrt{\alpha^{m+2}}C_{n,l}^{(m)}e^{il\chi-i\frac{l}{|l|}\chi}\left[\frac{n!\left(\frac{\beta_0}{2}\right)^{|l|+1}}{\pi(n+|l|)!}\right]^{\frac{1}{2}}\sqrt{\alpha^{|l|}}(n+1)r^{|l|-1}L_{n+1}^{|l|-1}e^{-\frac{\alpha\beta_0 r^2}{2}}\\
&\quad +\sum_{n=0}^{\infty}\sum_{|l|>0}\delta_{m,2n+|l|}\lambda^{m+1}\sqrt{\alpha^{m+2}}C_{n,l}^{(m)}e^{il\chi+i\frac{l}{|l|}\chi}\left[\frac{n!\left(\frac{\beta_0}{2}\right)^{|l|+1}}{\pi(n+|l|)!}\right]^{\frac{1}{2}}\sqrt{\alpha^{|l|}}\frac{\alpha\beta_0 r^{|l|+1}}{2}L_n^{|l|+1}e^{-\frac{\alpha\beta_0 r^2}{2}}\\
&\quad +2\sum_{n=0}^{\infty}\lambda^{2n+1}\alpha^{n+1}C_{n,l}^{(2n)}\left(\frac{e^{i\chi}}{2}+\frac{e^{-i\chi}}{2}\right)\sqrt{\frac{\beta_0}{2\pi}}\frac{\alpha\beta_0 r}{2}L_n^1 e^{-\frac{\alpha\beta_0 r^2}{2}}\,.
\end{aligned}
\tag{22}
$$

Now we make a change on the dummy indices of the sums. In particular, for the first term we define

$$
l'=l-\frac{l}{|l|},\quad n'=n+1
$$

so that $|l'| = |l| - 1$ and the condition $2n + |l| = m$ becomes

$$2n' + |l'| = m + 1\,.$$

For the second term, we introduce instead

$$l'' = l + \frac{l}{|l|}, \quad n'' = n$$

so that $|l''| = |l| + 1$ and the condition $2n + |l| = m$ becomes

$$2n'' + |l''| = m + 1\,.$$

We find

$$\begin{aligned}
&-\lambda\cos\chi\partial_r + \lambda\frac{\sin\chi}{r}\partial_\chi P = \\
&= -\sum_{n'=1}^{\infty}\sum_{|l'|>0}\delta_{m+1,2n'+|l'|}\lambda^{m+1}\sqrt{\alpha^{m+|l'|+3}}C^{(m)}_{n'-1,|l'|+1}e^{il'\chi}\left[\frac{n'!\left(\frac{\beta_0}{2}\right)^{|l'|+1}}{\pi(n+|l'|)!}\right]^{\frac{1}{2}}\sqrt{\frac{n'\beta_0}{2}}r^{|l'|}L^{|l'|}_{n'}e^{-\frac{\alpha\beta_0 r^2}{2}} \\
&\quad -\sum_{n=0}^{\infty}\lambda^{2n+2}\alpha^{n+2}C^{(2n+1)}_{n,1}\frac{\beta_0^2\sqrt{n+1}}{2\sqrt{\pi}}L^0_{n+1}e^{-\frac{\alpha\beta_0 r^2}{2}} \\
&\quad +\sum_{n''=0}^{\infty}\sum_{|l''|>0}\delta_{m+1,2n''+|l''|}\lambda^{m+1}\sqrt{\alpha^{m+|l''|+3}}C^{(m)}_{n'',|l''|-1}e^{il''\chi}\left[\frac{n''!\left(\frac{\beta_0}{2}\right)^{|l''|+1}}{\pi(n''+|l''|)!}\right]^{\frac{1}{2}}\sqrt{\frac{(n''+|l''|)\beta_0}{2}}r^{|l''|}L^{|l''|}_{n''}e^{-\frac{\alpha\beta_0 r^2}{2}}\,.
\end{aligned} \tag{23}$$

The second term on the r.h.s. of the above equation comes from the $l = \pm 1$ contributions to the first term on the r.h.s. of Eq. (22). Both the second and the third term on the r.h.s. of that equation contribute to the third term on the r.h.s. of Eq. (23). We can drop all the prime symbols of the dummy indices and get, recalling Eqs. (1c) and (1d),

$$\begin{aligned}
&-\lambda\cos\chi\partial_r + \lambda\frac{\sin\chi}{r}\partial_\chi P = \\
&= \sum_{n=0}^{\infty}\lambda^{2n+2}\alpha^{n+2}C^{(2n+2)}_{n+1,0}\sqrt{2\beta_0(n+1)}\frac{\beta_0^2\sqrt{n+1}}{2\sqrt{\pi}}L^0_{n+1}e^{-\frac{\alpha\beta_0 r^2}{2}} \\
&\quad +\sum_{n=0}^{\infty}\sum_{|l|>0}\delta_{m+1,2n+|l|}\lambda^{m+1}\sqrt{\alpha^{m+3}}\left(C^{(m)}_{n,|l|-1}\sqrt{\frac{(n+|l|)\beta_0}{2}} - C^{(m)}_{n-1,|l|+1}\sqrt{\frac{n\beta_0}{2}}\right)\phi_{n,l} \\
&= \sum_{n'''=0}^{\infty}\lambda^{2n'''}\alpha^{n'''+1}C^{(2n''')}_{n''',0}\sqrt{2\beta_0 n'''}\frac{\beta_0^2\sqrt{n'''}}{2\sqrt{\pi}}L^0_{n'''}e^{-\frac{\alpha\beta_0 r^2}{2}} \\
&\quad +\sum_{n=0}^{\infty}\sum_{|l|>0}\delta_{m''',2n+|l|}\lambda^{m'''}\sqrt{\alpha^{m'''+2}}C^{(m''')}_{n,l}[\beta_0 m''' + l^2]\phi_{n,l}
\end{aligned} \tag{24}$$

where two last changes of indices, $n + 1 \to n'''$ for the first term in the r.h.s. and $m + 1 \to m'''$ for the second one, were made. By dropping the prime symbols and merging the two expressions we finally get

$$-\lambda\cos\chi\partial_r + \lambda\frac{\sin\chi}{r}\partial_\chi P = \sum_{m=0}^{\infty}\lambda^m\sqrt{\alpha^{m+2}}\sum_{2n+|l|=m}C^{(m)}_{n,l}\left[\beta_0 m + l^2\right]\phi\,. \tag{25}$$

Putting together Eqs. (14), (15), (16) and (25), one finally obtains

$$\begin{aligned}
&\sum_{m=0}^{\infty}\lambda^m\sqrt{\alpha^{m+2}}\sum_{2n+|l|=m}C^{(m)}_{n,l}(\beta_0)\phi_{n,l}(\sqrt{\alpha}r,\chi|\beta_0)\left[\frac{\dot\alpha}{2\alpha} - \beta + \alpha\beta_0\right]\left(l + 2 - \alpha\beta_0 r^2 - \alpha\beta_0 r^2\frac{L^{|l|+1}_{n-1}}{L^{|l|}_n}\right) + \\
&+\left[\frac{\dot\alpha}{2\alpha} + \frac{\dot\lambda}{\lambda} + \alpha\beta_0 - \beta_0\right]\sum_{m=0}^{\infty}m\lambda^m\sqrt{\alpha^{m+2}}\sum_{2n+|l|=m}C^{(m)}_{n,l}(\beta_0)\phi_{n,l}(\sqrt{\alpha}r,\chi|\beta_0) = 0\,.
\end{aligned} \tag{26}$$

For the above equation to hold for every choice of $r$ and $\chi$, Eq. (9) of the main text has to be satisfied.

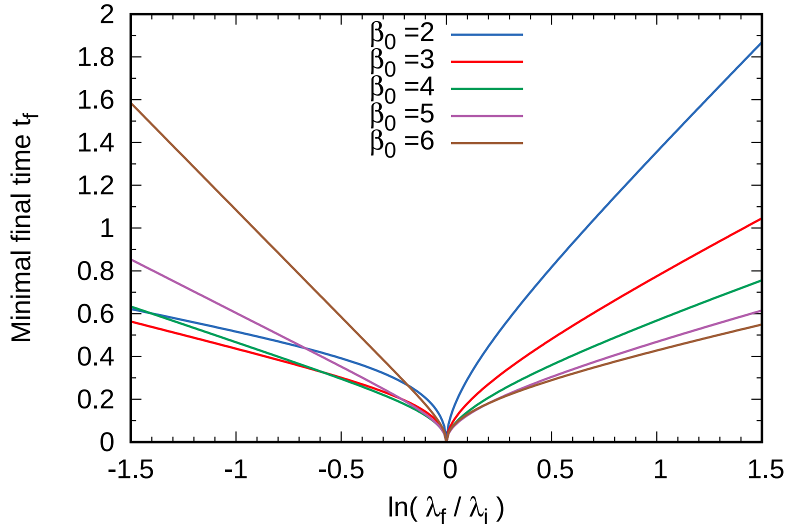


FIG. 1. Minimal time $t_f^{min}$ for the protocol studied in the main text as a function of the ratio $\lambda_f/\lambda_i$. Both increasing and decreasing activities are taken into account. Different colors correspond to different values of $\beta_0$. The constraint $1 \leq \beta(t) \leq 7$ is assumed.

## V. MINIMAL TIME

As discussed in the main text, the minimal time for the class of protocols studied can be computed by solving numerically Eqs. (19) and (20). In Fig. 1, we report the dependence of the solution on the boundary conditions for the control paramteres.

## VI. AVERAGE WORK

In this section, we compute the average work done by the external forces during the protocols described in the main text. The average work is defined as

$$\langle W \rangle = \int_0^{t_f} dt \int_0^{2\pi} d\chi \int_0^{\infty} dr\, r\, \partial_t U(r|\beta) P(r, \chi|\alpha, \lambda, \beta_0) = \frac{D_t}{\mu} \int_0^{t_f} dt \int_0^{2\pi} d\chi \int_0^{\infty} dr\, \dot{\beta} r^3\, P(r, \chi|\alpha, \lambda, \beta_0)\,, \tag{27}$$

where we have expressed the external potential in terms of the rescaled variables:

$$U = k\rho^2 = \frac{D_t \beta r^3}{\mu}\,. \tag{28}$$

By substituting the ansatz (12) in the above expression and recalling that

$$\int_0^{2\pi} d\chi e^{il\chi} = 2\pi \delta_{l,0} \tag{29}$$

we get

$$\langle W \rangle = \frac{D_t}{\mu} \int_0^{t_f} dt\, 2\pi \dot{\beta} \sum_{n=0}^{\infty} \lambda^{2n} \alpha^{n+1} C_{n,0}^{(2n)} \sqrt{\frac{\beta_0}{2\pi}} \int_0^{\infty} dr\, r^3 e^{-\frac{\alpha\beta_0 r^2}{2}} L_n^0 \left(\frac{\alpha\beta_0 r^2}{2}\right)\,. \tag{30}$$

By making use of Rodrigues formula Eq. (4), we can rewrite the last integral as

$$\begin{aligned}\int_0^\infty dr\, r^3 e^{-\frac{\alpha\beta_0 r^2}{2}} L_n^0\left(\frac{\alpha\beta_0 r^2}{2}\right) =& \frac{2}{\alpha^2\beta_0^2}\int_0^\infty dx\, x e^{-x} x L_n^0(x) \\ & \frac{2}{\alpha^2\beta_0^2}\int_0^\infty dx\, \frac{x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) \\ =& \frac{2}{\alpha^2\beta_0^2}\left(\delta_{n,0}-\delta_{n,1}\right)\,,\end{aligned} \tag{31}$$

leading to

$$\begin{aligned}\langle W\rangle &= \frac{D_t}{\mu}\int_0^{t_f} dt\, 2\pi\dot{\beta}\frac{2\sqrt{2\pi}}{\alpha\beta_0\sqrt{\beta_0}}\left(C_{0,0}^{(0)}-\frac{\lambda^2\alpha}{\sqrt{2}}C_{1,0}^{(2)}\right) \\ &= \frac{D_t}{\mu\beta_0}\int_0^{t_f} dt\,\dot{\beta}\left(\frac{2}{\alpha}+\frac{\lambda^2}{\sqrt{2}(\beta_0+1)}\right)\,.\end{aligned} \tag{32}$$

[1] K. Malakar, A. Das, A. Kundu, K. V. Kumar, and A. Dhar, Steady state of an active brownian particle in a two-dimensional harmonic trap, Physical Review E **101**, 022610 (2020).