Time scaling and quantum speed limit in non-Hermitian Hamiltonians

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We report on a time-scaling technique to enhance the performances of quantum control protocols in non-Hermitian systems. The considered time scaling involves no extra couplings and yields a significant enhancement of the quantum fidelity for a comparable amount of resources. We discuss the application of this technique to quantum-state transfers in two- and three-level open quantum systems. We derive the quantum speed limit in a system governed by a non-Hermitian Hamiltonian. Interestingly, we show that with an appropriate driving the time-scaling technique preserves the optimality of the quantum speed with respect to the quantum speed limit whereas reducing significantly the damping of the quantum-state norm.

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I. INTRODUCTION

Fast quantum control protocols have a promising future in quantum platforms as they mitigate the deleterious effects of disorder or/and dissipation. Introduced about a decade ago, shortcuts to adiabaticity (STA) [1] have already a strong track record of improving quantum procedures in a wide variety of quantum platforms, including ultracold atom setups [2–5], NMR [6], solid-state [7] systems, superconducting qubits [8], and topological spin chains [9,10]. There are several well-established methods to build STA protocols, such as the optimal control [11], the counterdiabatic driving [12–15], the use of Lewis-Riesenfeld invariants [16–20], or the use of properly scaled dynamical variables [21] to name a few. These methods provide different strategies to hamper, compensate, or mitigate the effects of nonadiabatic transitions.

As a matter of fact, STA protocols require more resources than adiabatic methods and may involve a larger number of dynamical couplings. For instance, the fast-forward technique as originally introduced by Masuda and Nakamura [22], Matsuda and Rice [23], Takahashi [24], and Zhu and Chen [25], requires the presence of extra couplings to be regularized in the limit of strong acceleration [26]. The same conclusion holds for most counterdiabatic protocols. Indeed, the dynamical control of additional interactions may constitute a limit for their practical implementation.

In the presence of dissipation, an arbitrary slow driving in the adiabatic limit is deleterious. The question is rather how to mitigate the effect of dissipation for a given protocol duration and to approach the ultimate limit provided by the quantum speed limit (QSL) [27–36] in non-Hermitian systems. In this article, we investigate a time-scaling method for finite Hilbert spaces that tackles these issues: It does not introduce extra couplings and preserves the quantum speed optimality with respect to the QSL in the considered systems whereas reducing the deleterious effect of the dissipation on the state norm. In addition, this strategy enables a minimization of the resources.

As a starting point, we consider a given quantum trajectory $|\psi_0(t)\rangle$ solution of the time-dependent Schrödinger equation for the Hamiltonian $\hat{H}(t)$,

$$i\hbar\frac{\partial|\psi_0(t)\rangle}{\partial t} = H(t)|\psi_0(t)\rangle, \tag{1}$$

where the time dependence of the Hamiltonian is encapsulated in a set of parameters $\hat{H}(t) = \hat{H}[\lambda_1(t), \dots, \lambda_N(t)]$. The quantum trajectory $|\psi_0[\Lambda(t)]\rangle$ is then a solution of the time-dependent Schrödinger equation for the rescaled Hamiltonian $\hat{H}^\Lambda(t)$,

$$\hat{H}^\Lambda(t) = \dot{\Lambda}\hat{H}\{\lambda_1[\Lambda(t)], \dots, \lambda_N[\Lambda(t)]\}. \tag{2}$$

where $\Lambda(t)$ is assumed to be a monotone differentiable function such that $\Lambda(0) = 0$ and $\dot{\Lambda}(t) \geqslant 0$ at any time.

The Hamiltonian (2) simply provides a time rescaling of the original solution. If T denotes the final time at which the system reaches the desired quantum-state target under the driving $\hat{H}(t)$, the evolution under the rescaled Hamiltonian $\hat{H}^\Lambda(t)$ reaches the very same target in a time $T^\Lambda = \Lambda^{-1}(T)$, that can be much shorter. As a result, time scaling provides *a priori* the simplest way to realize a shortcut to adiabaticity protocol.

The time-scaling method provides an enhancement of the protocol performance whereas maintaining the original quantum trajectory. In the following, we explain how to design the time-scaling $\Lambda(t)$ in a wide variety of contexts. The method is quite general and has a broad range of interdisciplinary applications in the increasingly relevant field of non-Hermitian quantum systems [37–39]. To work out quantitatively a strategy that minimizes the influence of dissipation, we define in Sec. II a driving that ensures a constant damping rate during the whole parametrized evolution. This systematic approach provides a clear improvement over the original driving and is illustrated in two- and three-level systems. This strategy can be applied jointly with geometric corrections on the driving field mitigating the effects of dissipation [40]. We explain how a suitable choice for the time-scaling function enables one to minimize the energetic cost of STA protocols for both closed and open systems whereas achieving the same quantum fidelity. In Sec. III, we discuss the relation between the time-scaling transform and the quantum speed limit. Generalizations of the QSL to open systems have been obtained within the density-matrix formalism [35], and in connection with the concept of Fisher information [36]. Here, we put forward a simple derivation of the QSL for quantum systems driven by non-Hermitian Hamiltonians in the spirit of the Vaidmann bound [30]. We show that the time-scaling transform preserves the ratio of the quantum speed to the QSL in two- and three-level dissipative systems with appropriate corrections to the quantum driving.

II. TIME SCALING FOR DISSIPATIVE TWO- AND THREE-LEVEL SYSTEMS

We discuss here the application of the time-scaling method to open quantum systems described by non-Hermitian Hamiltonian [41,42]. First, we address the commonly called fast quasiadiabatic (FAQUAD) protocol [43,44] for a dissipative two-level system [45]. We then investigate the application of time scaling to the stimulated Raman adiabatic passage (STIRAP) protocol in a three-level system. For each example, we first provide a reminder on a reference quantum protocol in the absence of dissipation. We, subsequently, show how the time scaling provides a systematic method to reach optimally the target state in the presence of dissipation.

A. FAQUAD driving in a two-level dissipative system

The FAQUAD protocol has been originally proposed for dissipationless quantum systems to perform a state to state transformation as quickly as possible whereas remaining as adiabatic as possible at all times. Let us recall the main features of this protocol for a two-level quantum system described by the control Hamiltonian,

$$\hat{H}_0(t) = \hbar \begin{pmatrix} \delta(t) & \Omega(t) \\ \Omega(t) & -\delta(t) \end{pmatrix} \tag{3}$$

expressed here in the decoupled $\{|e\rangle, |g\rangle\}$ basis with a real-valued Rabi frequency $\Omega(t)$. Its instantaneous eigenvalues are $E_\pm(t) = \pm\hbar\sqrt{\delta(t)^2 + \Omega(t)^2}$. They are associated with the instantaneous eigenvectors of $\hat{H}_0(t)$,

$$|\phi_+(\theta)\rangle = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{bmatrix}, \quad |\phi_-(\theta)\rangle = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) \\ -\cos\left(\frac{\theta}{2}\right) \end{bmatrix}, \tag{4}$$

where $\theta(t) = \text{acos}[\delta(t)/\sqrt{\delta(t)^2 + \Omega^2(t)}]$. In the FAQUAD approach, the time evolution of $\theta(t)$ is obtained by keeping constant the adiabatic criterium (equal to a constant c) [43],

$$\dot{\theta} = \frac{c}{\hbar} \frac{|E_+(t) - E_-(t)|}{|\langle\phi_+(\theta)|\partial_\theta\phi_-(\theta)\rangle|}, \tag{5}$$

where $0 < c < 1$. The $c \to 0$ limit is nothing but the adiabatic limit. In this protocol, nonadiabatic transitions are controlled through the "driving speed."

We now discuss the implementation of the time-scaling method on a FAQUAD driving protocol of control parameters noted $\{\Omega(t), \delta(t)\}$. This driving yields a quantum trajectory that follows closely quasiadiabatically the instantaneous eigenstate $|\phi_+[\theta(t)]\rangle$. By definition of a FAQUAD protocol, the angle $\theta(t)$ and the parameters $\{\Omega(t), \delta(t)\}$ fulfill the FAQUAD condition (5) for a given quasiadiabatic constant c. The time-scaling transform consists in applying instead the time-scaled control parameters $\{\Omega^\Lambda(t), \delta^\Lambda(t)\}$ with $\Omega^\Lambda(t) = \dot{\Lambda}(t)\Omega[\Lambda(t)]$ and $\delta^\Lambda(t) = \dot{\Lambda}(t)\delta[\Lambda(t)]$. These parameters yield a quantum trajectory $|\psi^\Lambda(t)\rangle = |\psi[\Lambda(t)]\rangle$ close to the instantaneous eigenvector $|\phi_+[\theta^\Lambda(t)]\rangle$ where $\theta^\Lambda(t) = \theta[\Lambda(t)]$. In the time-scaled protocol, the left-hand side of the FAQUAD condition (5) involves the rescaled time-dependent angle $\theta^\Lambda(t)$, whereas the right-hand side involves the eigenvalues of the rescaled Hamiltonian $\hat{H}^\Lambda(t) = \dot{\Lambda}(t)\hat{H}[\Lambda(t)]$. Both sides of Eq. (5) are, thus, multiplied by the same factor $\dot{\Lambda}(t)$ under the time-scaling transform, leaving the FAQUAD condition (5) and the constant c unaffected. As a result, we can use this invariance in order to engineer a scaling function $\Lambda(t)$ which fulfills extra requirements. We propose hereafter to set $\Lambda(t)$ by defining the acceptable dissipation rate for the considered quantum protocol.

We model the influence of dissipation with the following non-Hermitian Hamiltonian [45],

$$\hat{H}(t) = \hat{H}_0(t) - i\hbar\hat{\gamma}. \tag{6}$$

Originally introduced in nuclear physics [46], effective non-Hermitian Hamiltonians describe diverse phenomena, such as superradiance [47], non-Hermitian transport [38], or time crystals [39]. The dissipation operator $\hat{\gamma}$ typically depends on the system-environment coupling and on the environment quantum state and will be treated as a given parameter here. For the considered two-level system, we assume a diagonal dissipation operator $\hat{\gamma} = \gamma_e|e\rangle\langle e| + \gamma_g|g\rangle\langle g|$, where $|e\rangle$ ($|g\rangle$) denote the excited (ground) state. For the sake of generality, we use a generic two-level system, which may correspond to a two-level atom, to a quantum dot, or to a spin 1/2 depending on the context. The effective Hamiltonian (3) and (6) is relevant in a wide variety of domains [37] including NMR systems [48], quantum electrodynamics, quantum information, solid-state quantum dots, and quantum optics. Therefore, the presented time-scaling technique has a broad interdisciplinary range of applications.

We detail below a systematic procedure to choose the time-scaling $\Lambda(t)$ in order to mitigate dissipation effects in a FAQUAD driving. We note $\mathcal{N}^\Lambda(t) = \sqrt{\langle\psi^\Lambda(t)|\psi^\Lambda(t)\rangle}$ as the norm of the quantum state driven by the time-scaled control parameters $\{\Omega^\Lambda(t), \delta^\Lambda(t)\}$. As $|\psi^\Lambda(t)\rangle \approx \mathcal{N}^\Lambda(t)|\phi_+[\theta^\Lambda(t)]\rangle$, the corresponding norm $\mathcal{N}^\Lambda$ can be seen as a function of the angle θ parametrizing the trajectory. Otherwise stated, we define the norm $\mathcal{N}^\Lambda_\theta$ by $\mathcal{N}^\Lambda_\theta[\theta^\Lambda(t)] = \mathcal{N}^\Lambda(t)$. Taking advantage of the extra freedom provided by the time-scaling function, we impose a fixed "geometric" damping rate along the trajectory,

$$\frac{1}{\mathcal{N}^\Lambda_\theta} \frac{d\mathcal{N}^\Lambda_\theta}{d\theta} = -c', \tag{7}$$

with a given constant c' characterizing the quality of the driving. Equation (7) acts as a gauge-fixing condition determining the time-scaling $\Lambda(t)$ and can be easily integrated for the considered quantum trajectory $0 \leqslant \theta \leqslant \pi$ as $\mathcal{N}_\theta^\Lambda(\pi) = \mathcal{N}_\theta^\Lambda(0)\exp[-\pi c']$. For a given quantum trajectory, the final quantum-state damping, thus, depends only on the value of the constant c', which is determined by the desired quantum protocol reliability.

The evolution of the norm $\mathcal{N}^\Lambda$ of the rescaled quantum state under the quasiadiabatic approximation (see Appendix A) is given by

$$\frac{1}{\mathcal{N}^\Lambda}\frac{d\mathcal{N}^\Lambda}{dt} = -\langle\phi_+[\theta^\Lambda(t)]|\hat{\gamma}|\phi_+[\theta^\Lambda(t)]\rangle, \tag{8}$$

where we have substituted the eigenstate $|\phi_+[\theta^\Lambda(t)]\rangle$ to $|\psi^\Lambda(t)\rangle$ for the estimation of the instantaneous dissipation rate. This latter approximation is justified for the values of the constant c considered below. Besides, this approximation is only used for the design of the time-scaling $\Lambda(t)$—once we have this time scaling at hand, we evaluate its performance by resolving numerically the full Schrödinger equation. Using the chain rule,

$$\frac{d\mathcal{N}^\Lambda}{dt} = \frac{d\mathcal{N}_\theta^\Lambda}{d\theta^\Lambda}\frac{d\theta^\Lambda}{d\Lambda}\dot{\Lambda}, \tag{9}$$

and dividing Eq. (8) by Eq. (7), one obtains

$$\dot{\Lambda}\frac{d\theta^\Lambda}{d\Lambda} = \frac{1}{c'}\langle\phi_+(\theta^\Lambda)|\hat{\gamma}|\phi_+(\theta^\Lambda)\rangle. \tag{10}$$

This first-order differential equation with the initial condition $\Lambda(0) = 0$ enables a complete determination of the time-scaling $\Lambda(t)$ associated with a constant geometric damping. From Eq. (10), we see that the gauge-fixing condition (7) prescribes a driving speed $\dot{\theta}^\Lambda = \dot{\Lambda}\, d\theta^\Lambda/d\Lambda$ proportional to the instantaneous dissipation rate. It encodes mathematically the intuitive idea according to which one should increase the driving speed in a region of strong dissipation. Interestingly, the driving speed depends here only on geometric features of the trajectory $|\phi_+(\theta)\rangle$, i.e., on the orientation of the associated Bloch vector and not on its norm.

As a concrete example, we apply the time scaling to a given FAQUAD protocol [43,44] for which one has a constant Rabi frequency $\Omega(t) = \Omega_0$, a time-dependent detuning $\delta(t)$, and a given quasiadiabaticity constant c [see Eq. (5)]. The quantum trajectory is determined by the quasiadiabaticity condition (5) $\cos\theta(t) - \cos\theta_0 = -4c\Omega_0 t$ (see Appendix A). To ensure a high-fidelity transfer, the angle $\theta(t)$ must fulfill the boundary conditions $\theta_0 \simeq 0$ and $\theta_T \simeq \pi$. We choose $\cos\theta_0 = 1 - \epsilon$ and $\cos\theta_T = -1 + \epsilon$ with $0 < \epsilon \ll 1$ as a null parameter $\epsilon = 0$ generates unrealistic infinite detuning at the time boundaries [43,44]. The quasiadiabaticity constant is, thus, related to the protocol parameters as $c = (1-\epsilon)/(2\Omega_0 T)$. One readily finds the angle $\theta(t) = \arccos[f_\epsilon(t)]$ with $f_\epsilon(t) = (1-\epsilon)(1-2t/T)$, and the corresponding detuning $\delta(t) = \Omega/\tan\theta(t)$. We find $\langle\phi_+[\theta(t)]|\hat{\gamma}|\phi_+[\theta(t)]\rangle = \frac{1}{2}(\gamma_e + \gamma_g) + \frac{1}{2}(\gamma_e - \gamma_g)f_\epsilon(t)$ and $\dot{\theta}(t) = 2(1-\epsilon)/[T\sqrt{1 - f_\epsilon(t)^2}]$. The scaling $\Lambda(t)$ is, subsequently, obtained by solving (10). Here, we look for a time scaling keeping the same protocol duration, namely, $\Lambda(T) = T$.

In Fig. 1, we summarize the results of the original FAQUAD and of its time-scaled version for a specific example in the presence of dissipation. We have taken the

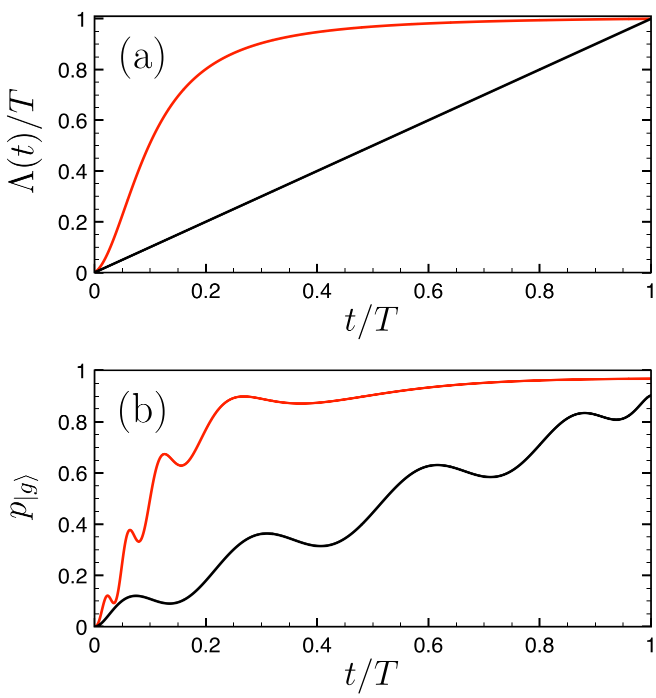

FIG. 1. Time-scaled FAQUAD protocol for a two-level system: (a) Time-scaling function $\Lambda(t)/T$ (solid red line) as a function of the renormalized time t/T. The solid black line corresponds to the original protocol [$\Lambda(t) = t$]. (b) Time-dependent occupation probability of the ground-state $|g\rangle$ for the original (solid black line) FAQUAD protocol and for its time-scaled version (solid red line) as a function of the renormalized time t/T. Parameters: $\Omega_0 T = 10$, $\epsilon = 0.01$, $\gamma_e T = 0.1$, and $\gamma_g = 0.01\gamma_e$.

parameters $\Omega_0 T = 10$ and $\epsilon = 0.01$, yielding an adiabaticity constant $c \simeq 0.05$. We consider the following dissipation rates: $\gamma_e T = 0.1$ and $\gamma_g = 0.01\gamma_e$. For our parameters, the condition $\Lambda(T) = T$ dictates the value of the constant $c' \simeq 5.3 \times 10^{-3}$. The purity is defined as the fraction of the target state population and reads $p(t) = p_{|g\rangle}(t)/[p_{|g\rangle}(t) + p_{|e\rangle}(t)]$ for the considered transfer to the ground state where $p_{|g\rangle}(t)$ [$p_{|e\rangle}(t)$] is the probability of occupation of the ground (excited) state at time t. Both protocols yield the same purity $p(T) \simeq 0.998$ at the final time. However, our time-scaled FAQUAD protocol yields a norm reduction $\mathcal{N}^\Lambda(T) \simeq 0.97$ to be compared to $\mathcal{N}(T) \simeq 0.90$ for the initial protocol. The time scaling, thus, significantly enhances the performance of the FAQUAD driving in the presence of dissipation by keeping a high purity whereas reducing the norm damping by, at least, a factor of 3 in this specific case.

B. Time scaling in a STIRAP transfer

1. The dissipationless STIRAP solution

In this section, we investigate the interest of time scaling for an accelerated population transfer in a dissipative three-level system. More precisely, we consider a three-level system in a Λ configuration, namely, a situation where the intermediate-state $|2\rangle$ has a higher energy than the initial and final states $|1\rangle, |3\rangle$. In the absence of dissipation, the quantum-state $|\psi(t)\rangle = C_1(t)|1\rangle + C_2(t)|2\rangle + C_3(t)|3\rangle$ obeys the Schrödinger equation associated with the control Hamiltonian,

$$\hat{H}_0(t) = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega_p^0(t) & 0 \\ \Omega_p^0(t) & 0 & \Omega_s^0(t) \\ 0 & \Omega_s^0(t) & 0 \end{pmatrix}, \tag{11}$$

where $\Omega_p^0(t)$ and $\Omega_s^0(t)$ are real valued and correspond to the pump and Stokes fields, respectively. This Schrödinger equation is formally equivalent to a precession equation for an effective spin $\mathbf{S}_0(t)$ defined in terms of the quantum-state components as $\mathbf{S}_0(t) = -C_3(t)\hat{x} - iC_2(t)\hat{\mathbf{y}} + C_1(t)\hat{\mathbf{z}}$ and with an effective magnetic-field $\mathbf{B}_0(t) = \frac{1}{2}[\Omega_p^0(t)\hat{\mathbf{x}} + \Omega_s^0(t)\hat{\mathbf{z}}]$ (see Appendix B),

$$\frac{d\mathbf{S}_0}{dt} = \mathbf{B}_0 \times \mathbf{S}_0. \tag{12}$$

The transfer of the population from initial-state $|1\rangle$ to final-state $|3\rangle$ can be realized by following adiabatically the dark state (associated with the zero eigenvalue), which amounts to applying Stokes and pump field pulses with a slight delay whereas maintaining a significant temporal overlap between the two pulses [49]. Using an invariant-based inverse engineering, such a transfer can be accelerated at the expense of a transient population in excited-state $|2\rangle$. In this latter protocol, the dissipation-free quantum trajectory follows the eigenvector of a Lewis-Riesenfeld invariant operator (Appendix C). Precisely, the quantum trajectory can be parametrized as [17]

$$|\psi_0(t)\rangle = \begin{pmatrix} \cos\gamma_0(t)\cos\beta_0(t) \\ -i\,\sin\gamma_0(t) \\ -\cos\gamma_0(t)\sin\beta_0(t) \end{pmatrix}. \tag{13}$$

For this quantum trajectory, the effective spin $\mathbf{S}_0(t)$ is real valued and reads $\mathbf{S}_0(t) = \cos\,\gamma_0(t)\sin\,\beta_0(t)\hat{\mathbf{x}} - \sin\,\gamma_0(t)\hat{\mathbf{y}} + \cos\,\gamma_0(t)\cos\,\beta_0(t)\hat{\mathbf{z}}$. In the absence of dissipation, we introduce a reference trajectory $\mathbf{S}_0(t)$ associated with a prescribed evolution of the angles $\beta_0(t)$ and $\gamma_0(t)$ that fulfills the required boundary conditions to ensure the transfer of the population from state $|1\rangle$ to state $|3\rangle$. The pump and Stokes fields $\Omega_{p0}(t), \Omega_{s0}(t)$ are, subsequently, inferred from the chosen trajectory (Appendix C).

2. The STIRAP solution in the presence of dissipation

We now take into account dissipation. We assume that the intermediate level $|2\rangle$ suffers a finite damping, modeled by the anti-Hermitian Hamiltonian $\hat{H}_\Gamma = -i\hbar\Gamma_2|2\rangle\langle 2|$. The effective spin now obeys the differential equation (Appendix B),

$$\frac{d\mathbf{S}}{dt} = \mathbf{B} \times \mathbf{S} - \overline{\overline{\Gamma}}\,\mathbf{S}, \tag{14}$$

where the dissipation tensor is $\overline{\overline{\Gamma}} = \Gamma_2\hat{y}\hat{y}$. By superimposing to the original field $\mathbf{B}_0(t)$ the following geometric correction [40]:

$$\delta\mathbf{B}_0(t) = \mathbf{S}_0(t) \times \overline{\overline{\Gamma}}\mathbf{S}_0(t), \tag{15}$$

the effective spin $\mathbf{S}$ follows the same trajectory as its dissipationless counterpart $\mathbf{S}_0$ despite the damping. Equivalently, the renormalized state $|\tilde{\psi}(t)\rangle = |\psi(t)\rangle/\|\,|\psi(t)\rangle\|$ coincides with its dissipationless counterpart. The corresponding pulse corrections read

$$\delta\Omega_p^0(t) = -\Gamma_2\sin 2\gamma_0(t)\cos\beta_0(t),$$
$$\delta\Omega_s^0(t) = \Gamma_2\sin 2\gamma_0(t)\sin\beta_0(t). \tag{16}$$

The effective spin $\mathbf{S}$ evolves in the magnetic-field $\mathbf{B}(t) = \mathbf{B}_0(t) + \delta\mathbf{B}_0(t)$. Interestingly, this approach restores the dissipation-free purity $p = p_{|3\rangle}/(p_{|1\rangle} + p_{|2\rangle} + p_{|3\rangle}) \simeq 99.8\%$ of the final population in the target state. However, and similar to the previous examples, the quantum-state norm $\mathcal{N}(t) = \|\,|\psi(t)\rangle\|$ suffers a damping in the course of the protocol. The time-scaling method enables one to mitigate this effect. For the STIRAP problem, one readily derives the rescaled pulse fields $\Omega_{p,s}^{0\,\Lambda}(t) = \dot{\Lambda}(t)\Omega_{p,s}^0[\Lambda(t)]$ and $\delta\Omega_{p,s}^{0\,\Lambda}(t) = \delta\Omega_{p,s}^0[\Lambda(t)]$. With the considered $\mathbf{S}_0$ trajectory, the population in the damped intermediate-state $p_{|2\rangle}(t) = |\langle 2|\psi(t)\rangle|^2 = \sin^2[\gamma_0(t)]$ reaches its maximum value at the half-time $t = T/2$.

3. Comparing different time-scaled STIRAP solutions

We propose hereafter two different time scalings that accelerate about this half-time to reduce the norm decrease. First, we consider a polynomial scaling that fulfills this latter requirement,

$$\Lambda_1(t) = T_1 P(t/T_1) \quad \text{with } P(x) = 3x^2 - 2x^3. \tag{17}$$

Alternatively, we will consider a time-scaling $\Lambda_2(t)$ [see Fig. 2(b)] associated with a uniform damping of the quantum-state norm in the sense of (7) and with respect to the geometric angle β_0. The scaling $\Lambda_2(t)$ is obtained by solving a differential equation analogous to (10) for the considered quantum system [with $\beta_0^\Lambda(\Lambda)$ replacing $\theta^\Lambda(\Lambda)$] to fix the condition (7),

$$\dot{\Lambda} = \frac{\langle\psi_0^\Lambda(t)|i\hat{H}_\Gamma|\psi_0^\Lambda(t)\rangle}{(d\beta_0^\Lambda/d\Lambda)c'}, \tag{18}$$

where $|\psi_0^\Lambda(t)\rangle = |\psi_0[\Lambda(t)]\rangle$, $\gamma_0^\Lambda(t) = \gamma_0[\Lambda(t)]$, and $\beta_0^\Lambda(t) = \beta_0[\Lambda(t)]$. The instantaneous dissipation is now proportional to $\langle\psi_0^\Lambda(t)|i\hat{H}_\Gamma|\psi_0^\Lambda(t)\rangle = \Gamma_2\sin^2\gamma_0^\Lambda(t)$. These three protocols: the pulse sequence $\Omega_{p,s}(t) = \Omega_{p,s}^0(t) + \delta\Omega_{p,s}^0(t)$ that compensates for dissipation and their time-scaled versions based on $\Lambda_1(t)$ and $\Lambda_2(t)$ are represented in Fig. 2.

For numerical applications, we use the angular trajectories detailed in Appendix C parametrized with $\epsilon = 0.05$, $\delta = \pi/4$, and for a damping rate equal to $\Gamma_2 T = 0.1$. First, we consider time-scaling $\Lambda_{1,2}(t)$ such that $\Lambda_{1,2}(T) = T$. This condition amounts to setting $T_1 = T$ and $c' \simeq 4.94 \times 10^{-3}$. One obtains the respective quantum fidelities $\mathcal{F}_0 = 0.954$, $\mathcal{F}_{\Lambda_1,T} = 0.966$, and $\mathcal{F}_{\Lambda_2,T} = 0.982$, respectively, for the initial protocol, for the polynomial scaling $\Lambda_1(t)$, and for the scaling $\Lambda_2(t)$. For the three protocols the final purity remains equal to the dissipation-free value of $p(T) \simeq 99.8\%$ [see Fig. 2(c)]. This quantity is defined in the STIRAP protocol as $p(T) = p_{|3\rangle}(T)/[p_{|1\rangle}(T) + p_{|2\rangle}(T) + p_{|3\rangle}(T)]$ with $p_{|n\rangle}(T)$ as the final probability of occupation of quantum-state $|n\rangle$. The enhancement of the quantum fidelity, thus, results from a reduction of the quantum-state norm damping.

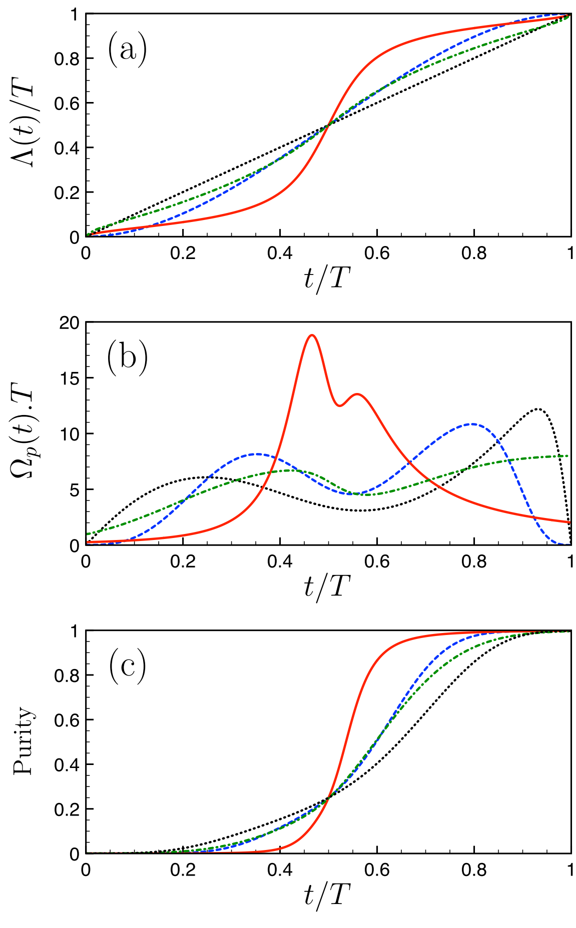

FIG. 2. Application of the time scaling to the STIRAP protocol: (a) Scaling functions $\Lambda_1(t)/T$ [Eq. (17), dashed blue line], $\Lambda_2(t)/T$ [Eq.(18), solid red line], and $\Lambda_3(t)/T$ [Eq. (19), dash-dotted green line] as a function of the normalized time t/T [with the choice $\Lambda_{1\text{-}3}(T) = T$]. The black dotted line represents the trivial time-scaling $\Lambda(t) = t$ and is associated with the original dissipation less protocol. (b) Rabi frequencies $\Omega_p(t)$ (in units of T^{-1}) for the pump field for the dissipation-corrected STIRAP protocols (16) and (C3) (dotted black line), and its time-scaled versions for $\Lambda_1(t)$ (dashed blue line), $\Lambda_2(t)$ (solid red line), and $\Lambda_3(t)$ (dot-dashed green line) as a function of the normalized time t/T. The original protocol corresponds to the angular trajectories (C4) with $\epsilon = 0.05$, $\delta = \pi/4$, and the dissipation rate is $\Gamma_2 = 0.1/T$. (c) Fraction of the population in the target state (purity) $p(t) = p_{|3\rangle}(t)/[p_{|1\rangle}(t) + p_{|2\rangle}(t) + p_{|3\rangle}(t)]$ as a function of the normalized time t/T for the dissipation-corrected STIRAP protocol (dotted black line), its time-scaled versions for $\Lambda_1(t)$ (dashed blue line), $\Lambda_2(t)$ (solid red line), and $\Lambda_3(t)$ (dashed-dot green line).

4. *Energetic cost and optimization*

Alternatively, one can choose the total duration $T_{1,2}$ of the time-scaling $\Lambda_{1,2}(t)$ as to yield a protocol with the same energy as the original STIRAP protocol. The energy, taken as $E^{\Lambda_k} = \int_0^{T_k} dt' \|\mathbf{B}^{\Lambda_k}(t')\|^2$, is proportional to the integrated Stokes and pump field intensities and inversely proportional to the total duration T^{Λ_k}. One finds the durations $T^{\Lambda_1} \simeq 1.10T$ and $T^{\Lambda_2} \simeq 1.53T$, giving the quantum fidelities $\mathcal{F}_{\Lambda_1, T^{\Lambda_1}} \simeq 0.963$ and $\mathcal{F}_{\Lambda_2, T^{\Lambda_2}} \simeq 0.974$. At constant energy and in this dissipative system, the time-scaling $\Lambda_2(t)$ applied to the original STIRAP protocol, thus, enables a reduction by nearly 45% of the discrepancy with a perfect transfer $\tilde{\epsilon} = 1 - \mathcal{F}$. This improvement at constant resources confirms the viability of the time-scaling technique.

The previous formalism provides a determination of a STIRAP transfer protocol with minimal energy whereas keeping the same quantum trajectory. First, we note that even for strong dissipation rates such that $\Gamma_\perp T = 1$ and $\Gamma_\parallel T = 0.1$ with the chosen angle $\epsilon = 0.05$, the energy overhead associated with the geometric correction (15) is negligible ($\delta E_{\rm corr}/E \simeq 0.6\%$). Regarding the optimization of the protocol through the time scaling, we can, thus, ignore this contribution and minimize $E[\Lambda, \dot{\Lambda}] = \hbar \int_0^T dt\, \dot{\Lambda}(t)^2 \{\Omega_p^0[\Lambda(t)]^2 + \Omega_s^0[\Lambda(t)]^2\}$ with the pump and Stokes fields (C3). By performing a Lagrangian minimization of this functional of Λ and $\dot{\Lambda}$, we obtain the following differential equation for the optimal scaling $\Lambda_3(t)$:

$$\dot{\Lambda} = c\left[\left(\frac{d\beta_0^\Lambda}{d\Lambda}\right)^2 \cot^2 \gamma_0^\Lambda(t) + \left(\frac{d\gamma_0^\Lambda}{d\Lambda}\right)^2\right]^{-1/2}, \qquad (19)$$

with the initial condition $\Lambda_3(0) = 0$. The constant c is determined self-consistently by imposing the boundary value $\Lambda_3(T)$. The solution of the differential equation obeyed by $\Lambda_3(t)$ imposes a constant norm for the effective field vector $\Omega^{\Lambda_3}(t) = \sqrt{\Omega_p^{\Lambda_3}(t)^2 + \Omega_s^{\Lambda_3}(t)^2} = cte$. This optimal solution is reminiscent of the π-pulse optimal solution for the two-level problem. Figures 2(a) and 2(b) represents this optimal time-scaling $\Lambda_3(t)$ [for $\Lambda_3(T) = T$] and the associated time-dependent Rabi frequency of the pump field. The time scaling accelerates when the effective magnetic field is minimal as, for instance, at the initial and final times. As a result of the constant norm of the effective field, the Stokes and pump fields no longer vanish at the initial and final times. With these optimal pulses, one obtains an energy $E_{\rm opt} \simeq 64.9\hbar/T$, which is roughly 10% lower than the original pulse $E_0 = 72.1\hbar/T$ for the same final purity $p(T) = 99.8\%$ [Fig. 2(c)].

III. TIME SCALING AND QUANTUM SPEED LIMIT FOR NON-HERMITIAN HAMILTONIANS

The time optimality of a quantum-state transfer is measured through the concept of QSL. Quantum systems evolve at a fraction of the QSL. This fraction constitutes a measure of the driving efficiency, and for an optimal driving, it reaches unity. One can readily show that this driving efficiency is invariant under a time-scaling transform for a unitary evolution. Indeed, for closed quantum system the time scaling changes equally the time and energy scales, respectively, related to the quantum speed and to the QSL. In contrast, the dissipation is unaffected by the time-scaling transform and one, therefore, expects a breakdown of driving efficiency for dissipative systems. In the following, we derive the expression of the QSL for dissipative systems modeled by a non-Hermitian Hamiltonian, and discuss its physical content on a dissipative two-level system. Surprisingly enough, for a three-level system, one can work out explicitly the quantum driving protocol that restores the dissipation-free driving efficiency. In this system, the quantum speed ratio to the QSL remains invariant through time scaling transforms despite the presence of dissipation.

A. Quantum speed limit for the non-Hermitian Hamiltonian

The QSL amounts to measuring the minimal time—associated with the maximal evolution velocity—from a given initial-state $|\tilde{\psi}(0)\rangle$ to a state orthogonal $|\tilde{\psi}(t)\rangle$ to the initial one [we denote $|\tilde{\psi}(t)\rangle = |\psi(t)\rangle/\||\psi(t)\rangle\|$]. For a system evolving under the action of a time-independent Hamiltonian $\hat{H}_0$, it translates as an upper bound on the rate of variation of the angle $\cos\phi = |\langle\psi(0)|\psi(t)\rangle|$,

$$\frac{d\phi}{dt} \leqslant \frac{\Delta\hat{H}_0}{\hbar}. \tag{20}$$

In Appendix D, we show how Vaidman's derivation of the QSL in dissipationless system [30] can be transposed to non-Hermitian time-dependent Hamiltonians $\hat{H} = \hat{H}_0 - i\hat{\Gamma}$. The new bound reads

$$\dot{\phi} = \frac{d\phi}{dt} \leqslant \frac{\sqrt{(\Delta\hat{H}_0)^2 + (\Delta\hat{\Gamma})^2 - i\langle[\hat{H}_0, \hat{\Gamma}]\rangle}}{\hbar}. \tag{21}$$

Our expression highlights the role of the dissipation operator (and its variance) in the evolution of the quantum state. As an example, this inequality shows that a strictly positive quantum speed limit $\chi(t) > 0$ exists for an otherwise stationary eigenstate of the Hermitian Hamiltonian ($\Delta\hat{H}_0 = 0$) as long as dissipation has a strictly positive variance $\Delta\hat{\Gamma} > 0$. As a consistency check, we have performed numerical simulations in two-level systems that confirm the validity of this upper bound and evidence the role played by the non-Hermitian terms in this dissipative QSL.

B. Quantum speed limit in a two-level system

We illustrate the application of the non-Hermitian QSL in a simple dissipative two-level quantum system driven by a resonant Rabi pulse. We discuss, in particular, how the QSL is affected by dissipation. To fix ideas, we consider a two-level atom $\{|e\rangle, |g\rangle\}$ driven by a resonant Rabi laser pulse, but our argument applies equally well to other generic two-level systems, such as a spin-1/2 driven by a magnetic field.

The renormalized quantum-state $|\widetilde{\psi}(t)\rangle = |\psi(t)\rangle/\sqrt{\langle\psi(t)|\psi(t)\rangle}$ is noted $|\widetilde{\psi}(t)\rangle = a_t|e\rangle + b_t|g\rangle$, and we assume the atom initially in the excited state, namely, $|\widetilde{\psi}(0)\rangle = |e\rangle$ (the quantum speed and the QSL are naturally invariant with respect to the multiplication of the initial state by a global phase). We use the parametrization $a_t = \cos(\theta_t/2)e^{i\varphi_{a,t}}$ and $b_t = \sin(\theta_t/2)e^{i\varphi_{b,t}}$. As shown in Appendix D, a necessary condition to saturate the QSL is that $\langle\dot{\widetilde{\psi}}(t)|\widetilde{\psi}(0)\rangle\langle\widetilde{\psi}(0)|\dot{\widetilde{\psi}}(t)\rangle$ be real valued at all times with $|\dot{\widetilde{\psi}}(t)\rangle$ the time derivative of the renormalized quantum state. This condition reads $|a_t|^2\dot{\varphi}_{a,t} = 0$ or simply $a_t \in \mathbb{R}$ and is equivalent to impose a planar trajectory for the Bloch vector representing the quantum-state $|\widetilde{\psi}(t)\rangle$.

The considered control Hamiltonian is time independent in the rotating frame and reads $\hat{H}_0 = \frac{1}{2}\Omega_0(|e\rangle\langle g| + |g\rangle\langle e|)$ with a Rabi frequency $\Omega_0 = \pi/T$. The dissipation is described by the operator $\hat{\Gamma} = \Gamma_1|e\rangle\langle e| + \Gamma_2|g\rangle\langle g|$, reflecting the decay rates of each atomic level, and the total non-Hermitian Hamiltonian reads $\hat{H} = \hat{H}_0 - i\hat{\Gamma}$ driving the quantum-state $|\psi_t\rangle$. The Schrödinger equation yields a coefficient $a_t \in \mathbb{R}$ at all times so that the necessary condition to saturate the QSL is fulfilled. Deriving the QSL [Eq. (21)] is a straightforward task, and one obtains $(\Delta H_0)^2(t) = \frac{1}{4}\Omega_0^2$, $(\Delta\hat{\Gamma})^2(t) = \Gamma_1^2|a_t|^2 + \Gamma_2^2|b_t|^2 - (\Gamma_1|a_t|^2 + \Gamma_2|b_t|^2)^2$, and $i\langle[\hat{H}_0, \hat{\Gamma}]\rangle(t) = \frac{i}{4}(\Gamma_1 - \Gamma_2)\Omega_0(a_t b_t^* - a_t^* b_t)$. One finds that the quantum speed is equal to the QSL.

To clarify how dissipation affects the quantum speed, we consider in the following two opposite cases: $\Gamma_e > \Gamma_g$ and $\Gamma_e < \Gamma_g$. In the first configuration, the faster decay of the excited state contributes to flip down the Bloch vector. One thus expects a quantum velocity faster than in the dissipation-free case. In the opposite situation ($\Gamma_e < \Gamma_g$), the ground state is less stable, and one expects dissipation to slow down the quantum-state transfer. Our expression for the QSL (D3) captures this physics through the commutator $i\langle[\hat{H}_0(t), \hat{\Gamma}]\rangle$: depending on the relative strength of the excited- and ground-state dissipation rates, this contribution increases or decreases the QSL.

As an example, with the dissipation rates $\Gamma_e T = 0.2$ and $\Gamma_g T = 0.01$, the quantum state evolves faster than in the dissipation-free system for $\Gamma_e > \Gamma_g$, and the π pulse is achieved for $T^* \simeq 0.96T$ whereas $T^* \simeq 1.04T$ when the values of the dissipation rates are exchanged. In both cases the damping seriously deteriorates the quality of the transfer and the final quantum fidelity. In these examples, the quantum speed reaches the QSL at all times. Such a saturation of the QSL persists after a time-scaling transform. More generally, we show below that the time-scaling transform can also preserve the ratio of the quantum speed to the QSL in a dissipative three-dimensional system.

C. Quantum speed limit in a dissipative STIRAP system

We now consider the dissipative three-level system of Sec. II B. We use the pulse sequence $\Omega_{p,s}(t) = \Omega^0_{p,s}(t) + \delta\Omega^0_{p,s}$ corresponding to the sum of the dissipation-free pulses $\Omega^0_{p,s}(t)$ (C3) and the associated geometric corrections $\delta\Omega^0_{p,s}(t)$ (16). Thanks to the pulse correction, the renormalized quantum-state $|\widetilde{\psi}(t)\rangle$ follows exactly the dissipation-free trajectory, i.e., $|\widetilde{\psi}(t)\rangle = |\psi_0(t)\rangle$ at all times. Thus, the angle $\phi(t) = \arccos[|\langle\widetilde{\psi}(t)|\widetilde{\psi}(0)\rangle|]$ fulfills $\phi(t) = \phi_0(t)$ at all times, where $\phi_0(t) = \arccos[|\langle\psi_0(t)|\psi(0)\rangle|]$ is the angle associated with the dissipation-free trajectory. The effective quantum speed $\dot{\phi}(t)$ is, thus, given by the dissipation-free trajectory.

In the corrected protocol, the non-Hermitian QSL $\chi(t)$ depends *a priori* on the dissipation-free control Hamiltonian $\hat{H}_0$, the geometric correction $\delta\hat{H}_0$, and the dissipation operator $\hat{\Gamma}$. We find $\chi(t)^2 = (\Delta\hat{H}_0)^2 + \langle\{\delta\hat{H}_0, \hat{H}_0\}\rangle + (\Delta\delta\hat{H}_0)^2 + (\Delta\hat{\Gamma})^2 - i\langle[\hat{H}_0, \hat{\Gamma}]\rangle - i\langle[\delta\hat{H}_0, \hat{\Gamma}]\rangle$. Remarkably, by using the explicit form of the geometric correction (16), the dissipative QSL $\chi(t)$ boils down to $\chi(t) = \Delta\hat{H}_0(t)$. Thanks to the geometric correction, the non-Hermitian QSL (21) is exactly equal to the dissipation-free QSL (D7) of the original protocol. The preservation of the dissipation-free quantum speed $\dot{\phi}_0(t)$ and QSL $\chi_0(t)$ despite dissipation comes at the price of an energy overhead associated to the extra term added to the Hamiltonian $\delta\hat{H}_0$.

As a corollary, when the time-scaling technique is applied, both the quantum speed and the dissipative QSL undergo similar transformations as $\dot{\phi}^\Lambda(t) = \dot{\Lambda}(t)\dot{\phi}_0[\Lambda(t)]$ and $\chi^\Lambda(t) = \dot{\Lambda}(t)\chi_0[\Lambda(t)]$. That is to say, the time scaling preserves the ratio $r_0(t)$ of the quantum speed to the QSL as $r^\Lambda(t) = r_0[\Lambda(t)]$ with $r_0(t) = \hbar\dot{\phi}_0(t)/\dot{\chi}_0(t)$. Remarkably and thanks to the geometric correction, the dissipative dynamics keeps the same quantum speed and quantum speed limit as for the original dissipation-free protocol.

IV. CONCLUSION

In conclusion, we have demonstrated the applicability and relevance of the time-scaling method for quantum-state transfer as a tool to optimize the resources and/or mitigate the effect of dissipation in non-Hermitian quantum systems. Actually, the time-scaling function provides an extra freedom in the system that can be used to minimize the energy used in the protocol or the norm reduction of the quantum state. This strategy yields a significant enhancement of the performance of widely employed protocols in simple quantum systems (two- and three-level systems), such as the STIRAP transfer. In particular, it provides a mitigation of dissipation that can be strongly beneficial in quantum architectures performing successive operations on quantum states of small dimension. The quantum speed limit has been here generalized to non-Hermitian Hamiltonians, and we have shown that time scaling does not affect the ratio of the quantum speed to the quantum speed limit. In particular, the optimality is preserved when the system is driven at the quantum speed limit. Perspectives for this paper include the application of the time-scaling technique to the phenomena of non-Hermitian transport.

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APPENDIX A: DETERMINATION OF THE SCALING IN A FAQUAD PROTOCOL

We derive hereafter the FAQUAD quantum trajectory $\theta(t)$ by resolving Eq. (5). From the expression of the eigenvectors [see Eq. (4)] and of the associated energies, one finds $\langle\phi_+(\theta)|\partial_\theta\phi_-(\theta)\rangle = \frac{1}{2}$ and $|E_+(t) - E_-(t)| = 2\hbar\sqrt{\delta(t)^2 + \Omega(t)^2}$. Using the relation $\sin\theta(t) = \Omega(t)/\sqrt{\delta(t)^2 + \Omega(t)^2}$, Eq. (5) can be recast as

$$d\theta\,\sin\theta = 4c\,\Omega(t)dt. \tag{A1}$$

For a constant Rabi pulse $\Omega(t) = \Omega_0$, it is readily integrated as $\cos\theta(t) - \cos\,\theta_0 = -4c\Omega_0 t$.

To establish Eq. (8) for the quantum-state norm, we first use the non-Hermitian Schrödinger equation for the time-scaled evolution,

$$i\hbar\frac{d|\psi^\Lambda(t)\rangle}{dt} = \{\dot{\Lambda}(t)\hat{H}_0[\Lambda(t)] - i\hbar\hat{\gamma}\}|\psi^\Lambda(t)\rangle.$$

Combining this equation with its Hermitian conjugate, one deduces the equation obeyed by the quantum-state norm,

$$\frac{d[\mathcal{N}^\Lambda(t)^2]}{dt} = \frac{d\langle\psi^\Lambda(t)|\psi^\Lambda(t)\rangle}{dt} = -2\langle\psi^\Lambda(t)|\hat{\gamma}|\psi^\Lambda(t)\rangle.$$

We then substitute $|\psi^\Lambda(t)\rangle$ by its quasiadiabatic approximation $|\psi^\Lambda(t)\rangle \approx \mathcal{N}^\Lambda(t)|\phi_+[\theta^\Lambda(t)]\rangle$.

APPENDIX B: FORMULATION OF THE STIRAP DYNAMICS AS THE PRECESSION OF AN EFFECTIVE SPIN

The equivalence between the Schrödinger equation driven by the STIRAP Hamiltonian (3) and the precession of an effective spin $\mathbf{S}_0(t)$ is a well-known result [49]. We recall here the corresponding steps to ease the reading. The Schrödinger equation reads

$$\begin{aligned}\frac{dC_1(t)}{dt} &= \frac{1}{2}\Omega_p(t)[-iC_2(t)],\\ \frac{d[-iC_2(t)]}{dt} &= -\frac{1}{2}\Omega_p(t)C_1(t) - \frac{1}{2}\Omega_s(t)C_3(t),\\ \frac{d[-C_3(t)]}{dt} &= \frac{1}{2}\Omega_s(t)[-iC_2(t)],\end{aligned} \tag{B1}$$

which corresponds to Eq. (14) with the effective spin $\mathbf{S}_0(t) = -C_3(t)\hat{x} - iC_2(t)\hat{\mathbf{y}} + C_1(t)\hat{\mathbf{z}}$ and the effective magnetic-field $\mathbf{B}_0(t) = \frac{1}{2}[\Omega_p^0(t)\hat{\mathbf{x}} + \Omega_s^0(t)\hat{\mathbf{z}}]$. The presence of an additional anti-Hermitian contribution $\hat{H}_\Gamma = -i\hbar\Gamma_2|2\rangle\langle 2|$ to the Hamiltonian results in an extra dissipative term $-\overline{\overline{\Gamma}}\,\mathbf{S}_0$ on the right-hand side of the precession equation, with the tensor $\overline{\overline{\Gamma}} = \Gamma_2\hat{y}\hat{y}$. This explains the form of the dissipative precession Eq. (14).

APPENDIX C: ACCELERATED STIRAP PROTOCOL

We reproduce for convenience the invariant-based [18] procedure used in Ref. [17] to obtain the accelerated STIRAP protocol. By definition, an invariant is an operator $\hat{I}(t)$ that fulfills the equation of motion,

$$\frac{\partial\hat{I}(t)}{\partial t} + \frac{1}{i\hbar}[\hat{I}(t), \hat{H}_0(t)] = 0. \tag{C1}$$

To obtain the explicit form of such an invariant, we express the Hamiltonian $\hat{H}_0(t)$ (11) in terms of the angular momentum operators for the spin 1, namely, $\hat{H}_0(t) = \frac{\hbar}{2}[\Omega_p^0(t)\hat{K}_1 + \Omega_s^0(t)\hat{K}_2]$ with the three operators $\hat{K}_1$, $\hat{K}_2$, $\hat{K}_3$ defined as

$$\hat{K}_1 = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \hat{K}_2 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

$$\hat{K}_3 = \begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}.$$

These operators fulfill the angular momentum commutation relations $[\hat{K}_1, \hat{K}_2] = i\hat{K}_3$, $[\hat{K}_2, \hat{K}_3] = i\hat{K}_1$ and $[\hat{K}_3, \hat{K}_1] = i\hat{K}_2$. We take the following ansatz $\hat{I}(t) = \cos\gamma_0(t)\sin\beta_0(t)\hat{K}_1 + \cos\gamma_0(t)\cos\beta_0(t)\hat{K}_2 + \sin\gamma_0(t)\hat{K}_3$. By virtue of the previous commutation relations, the operator $\hat{I}(t)$ follows Eq. (C1) if and only if,

$$\dot{\gamma}_0(t) = \tfrac{1}{2}\big[\Omega_p^0(t)\cos\beta_0(t) - \Omega_s^0(t)\sin\beta_0(t)\big],$$
$$\dot{\beta}_0(t) = \tfrac{1}{2}\tan\gamma_0(t)\big[\Omega_s^0\cos\beta_0(t) + \Omega_p^0(t)\sin\beta_0(t)\big]. \quad \text{(C2)}$$

These relations can be easily inverted in order to infer the pump and Stokes fields $\Omega_{p0}(t), \Omega_{s0}(t)$ from the invariant parameters,

$$\Omega_p^0(t) = 2\left(\dot{\beta}_0(t)\frac{\sin\beta_0(t)}{\tan\gamma_0(t)} + \dot{\gamma}_0(t)\cos\beta_0(t)\right),$$
$$\Omega_s^0(t) = 2\left(\dot{\beta}_0(t)\frac{\cos\beta_0(t)}{\tan\gamma_0(t)} - \dot{\gamma}_0(t)\sin\beta_0(t)\right). \quad \text{(C3)}$$

We now recall an essential property of Lewis-Riesenfeld invariants [16], which is a direct consequence of Eq. (C1). Consider the instantaneous eigenmode basis $\{|\phi_n(t)\rangle\}$ of the invariant operator $\hat{I}(t)$ and assume that the initial quantum state is expressed on these modes as $|\psi_0(0)\rangle = \sum_n c_n^0|\phi_n(0)\rangle$. Then, at any later time t, the quantum-state $|\psi_0(t)\rangle$ driven by the Hamiltonian $\hat{H}_0(t)$ reads $|\psi_0(t)\rangle = \sum_n c_n^0 e^{i\varphi_n(t)}|\phi_n(t)\rangle$. The specific expression of the phase $\varphi_n(t)$ is not needed here. Rather, we use the particular case in which the initial quantum-state $|\psi_0(0)\rangle$ is an eigenmode of the invariant $\hat{I}(0)$. Precisely, we choose the initial angular parameters $\gamma_0(0), \beta_0(0)$ in such a way that $|\psi_0(0)\rangle = e^{i\varphi_0}|\phi_0(0)\rangle$ for some phase φ_0 and where $|\phi_0(0)\rangle$ is the zero eigenmode of the initial invariant $\hat{I}(0)$. A diagonalization of $\hat{I}(t)$ shows that the eigenmode $|\phi_0(t)\rangle$ is given by the right-hand side of Eq. (13). The quantum-state $|\psi_0(t)\rangle$ then follows the instantaneous zero eigenmode and is also given by Eq. (13). We emphasize that there is no adiabaticity assumption here: This property is independent of the driving speed, which makes the Lewis-Riesenfeld technique particularly well suited for the implementation of shortcuts to adiabaticity.

We consider a polynomial form for the angular functions $\gamma(t) = \sum_{j=0}^4 a_j t^j$ and $\beta(t) = \sum_{j=0}^3 b_j t^j$. They fulfill the boundary conditions of Protocol 2 of Ref. [17],

$$\begin{aligned}
&\gamma_0(0) = \epsilon, \quad \dot{\gamma}_0(0) = 0, \quad \gamma_0(T/2) = \delta,\\
&\gamma_0(T) = \epsilon, \quad \dot{\gamma}_0(T) = 0,\\
&\beta_0(0) = 0, \quad \beta_0(T) = \pi/2,\\
&\dot{\beta}_0(0) = 0, \quad \dot{\beta}_0(T) = 0, \quad \gamma_0(T/2) = \delta.
\end{aligned} \quad \text{(C4)}$$

Such shortcut-to-adiabaticity solutions give rise to a trade-off between the amplitudes of Rabi frequencies and the transient population of intermediate-state $|2\rangle$ [17,50].

A small angle initial angle ϵ is used, which yields an error $1 - \mathcal{F} = O(\epsilon^2)$ for the protocol defined by Eqs. (C3) and (C4) alone. As in Ref. [17], the invariant technique is resilient to a small mismatch between the initial quantum state and the invariant eigenmode. For sake of simplicity, in our discussion on the quantum fidelity and of the quantum speed limit, we consider this protocol as such. However, a perfect transfer may be restored by adding an initial and a final stage to the STIRAP protocol (C4), namely, by using an initial and final small pulse of angle ϵ with the pump field $\Omega_p(t)$/Stokes field $\Omega_s(t)$ used separately. The full protocol would then correspond to a sequence $|1\rangle \to (\cos\epsilon|1\rangle - i\sin\epsilon|2\rangle) \to (-i\sin\epsilon|2\rangle + \cos\epsilon|3\rangle) \to |3\rangle$.

APPENDIX D: QUANTUM SPEED LIMIT FOR THE NON-HERMITIAN HAMILTONIAN

As a starting point, we recall that for any Hermitian operator $\hat{A}$ and any quantum-state $|\psi\rangle$ [30],

$$A|\psi\rangle = \langle A\rangle|\psi\rangle + \Delta A|\psi_\perp\rangle, \quad \text{(D1)}$$

where $|\psi_\perp\rangle$ is orthogonal to $|\psi\rangle$ and $\Delta\hat{A}$ is the variance of the operator $\hat{A}$.

Interestingly, this relation can be generalized to non-Hermitian Hamiltonian $\hat{H} = \hat{H}_0 - i\hat{\Gamma}$ (with the Hermitian operators $\hat{H}_0^\dagger = \hat{H}_0$ and $\hat{\Gamma}^\dagger = \hat{\Gamma}$) on a given normalized quantum-state $|\psi\rangle$. With the same notations as previously, and for any quantum-state $|\psi\rangle$, we write

$$\chi|\psi_\perp\rangle = \hat{H}|\psi\rangle - \langle\hat{H}\rangle|\psi\rangle, \quad \text{(D2)}$$

where χ is a positive real scalar (see below). To get an explicit expression for the coefficient χ, we write $\langle\hat{H}^\dagger\hat{H}\rangle = |\langle\hat{H}\rangle|^2 + \chi^2$ and use $\hat{H}^\dagger\hat{H} = \hat{H}_0^2 + \hat{\Gamma}^2 - i[\hat{H}_0, \hat{\Gamma}]$. As a result, we find

$$\chi = [(\Delta\hat{H}_0)^2 + (\Delta\hat{\Gamma})^2 - i\langle[\hat{H}_0, \hat{\Gamma}]\rangle]^{1/2}. \quad \text{(D3)}$$

The anti-Hermiticity of the commutator $[\hat{H}_0, \hat{\Gamma}]$ guarantees that the quantity $i\langle[\hat{H}_0, \hat{\Gamma}]\rangle$ is real valued.

In closed quantum systems, the usual definition of the quantum velocity rests on the fidelity with respect to initial-state $\mathcal{F}(t) = |\langle\psi(t)|\psi(0)\rangle|^2$—the quantum velocity is inversely proportional to the time for which this fidelity goes to zero. By using the decomposition (D2) in the Schrödinger equation, one obtains the time derivative of the fidelity for nonunitary dynamics,

$$\begin{aligned}
\dot{\mathcal{F}}(t) &= -2\langle\hat{\Gamma}\rangle(t)|\langle\psi(t)|\psi(0)\rangle|^2\\
&\quad -\frac{2\chi(t)}{\hbar}\mathrm{Re}[i\langle\psi(t)|\psi(0)\rangle\langle\psi(0)|\psi_\perp(t)\rangle]\\
&= \dot{\mathcal{F}}_r + \dot{\mathcal{F}}_\theta.
\end{aligned} \quad \text{(D4)}$$

The right-hand side has two contributions with distinct physical interpretations. The first component $\dot{\mathcal{F}}_r = -2\langle\hat{\Gamma}\rangle(t)|\langle\psi(t)|\psi(0)\rangle|^2$ corresponds to a pure quantum-state damping. In contrast, the contribution $\dot{\mathcal{F}}_\theta$ accounts for a genuine rotation of the quantum state. $\dot{\mathcal{F}}_\theta$ is, thus, the only relevant contribution to the quantum velocity.

We now propose a definition of the quantum velocity unaffected by the trivial quantum-state damping. For this purpose, we introduce the renormalized quantum state $|\widetilde{\psi}(t)\rangle = |\psi(t)\rangle/\sqrt{\langle\psi(t)|\psi(t)\rangle}$ and consider the corresponding quantum fidelity $\widetilde{\mathcal{F}}(t) = |\langle\widetilde{\psi}(t)|\widetilde{\psi}(0)\rangle|^2$. By construction, only the relevant angular velocity $\dot{\mathcal{F}}_\theta$ contributes to the variation of this quantum fidelity, i.e., $\dot{\widetilde{\mathcal{F}}}(t) = \dot{\widetilde{\mathcal{F}}}_\theta$.

To determine an upper bound on $\dot{\widetilde{\mathcal{F}}}_\theta$, we apply the concepts introduced in Ref. [30]. The initial state can always be expanded over at most three orthogonal states as $|\psi(0)\rangle = \langle\widetilde{\psi}(t)|\widetilde{\psi}(0)\rangle|\widetilde{\psi}(t)\rangle + \langle\widetilde{\psi}_\perp(t)|\widetilde{\psi}(0)\rangle|\widetilde{\psi}_\perp(t)\rangle + \alpha|\widetilde{\psi}_{\perp\perp}(t)\rangle$. This guarantees that $|\langle\widetilde{\psi}(0)|\widetilde{\psi}_\perp(t)\rangle| \leqslant \sqrt{1-|\langle\widetilde{\psi}(0)|\widetilde{\psi}(t)\rangle|^2}$. As a result,

$$|\dot{\widetilde{\mathcal{F}}}| = |\dot{\widetilde{\mathcal{F}}}_\theta| \leqslant \frac{2\chi(t)}{\hbar}|\langle\widetilde{\psi}(0)|\widetilde{\psi}(t)\rangle|\sqrt{1-|\langle\widetilde{\psi}(0)|\widetilde{\psi}(t)\rangle|^2}. \tag{D5}$$

By introducing the usual definition $\cos\phi = \widetilde{\mathcal{F}}^{1/2} = |\langle\widetilde{\psi}(t)|\widetilde{\psi}(0)\rangle|$, we obtain the following upper bound for the quantum velocity,

$$\dot{\phi}(t) \leqslant \frac{\sqrt{(\Delta\hat{H}_0)^2 + (\Delta\hat{\Gamma})^2 - i\langle[\hat{H}_0,\hat{\Gamma}]\rangle}}{\hbar}. \tag{D6}$$

From Eqs. (D4) and (D5), a necessary condition to saturate the QSL is to satisfy at all times $\langle\tilde{\psi}(t)|\tilde{\psi}(0)\rangle\langle\tilde{\psi}(0)|\tilde{\psi}_\perp(t)\rangle \in i\mathbb{R}$, or, equivalently, $\langle\widetilde{\psi}(t)|\widetilde{\psi}(0)\rangle\langle\widetilde{\psi}(0)|\dot{\widetilde{\psi}}(t)\rangle \in \mathbb{R}$, where $|\dot{\widetilde{\psi}}(t)\rangle$ denotes the time derivative of the normalized quantum state.

Other derivations of the QSL for dissipative systems, based on a matrix density formalism, can be found in Refs. [35,36]. Our nonunitary QSL has a similar form as the QSL obtained for closed quantum systems [30],

$$\dot{\phi}(t) \leqslant \frac{\Delta\hat{H}_0}{\hbar}, \tag{D7}$$

up to a replacement of the energy variance $\Delta\hat{H}_0$ by the quantity $\chi(t)$ (D3). By Ehrenfest's theorem, the variance $\Delta\hat{H}_0$ is time independent for a unitary evolution in a constant Hamiltonian, leading to a constant QSL in this context. Nevertheless, for the time-dependent and non-Hermitian Hamiltonians considered here, the QSL $\chi(t)$ generally varies with time. By construction the quantity $\chi^2 = (\Delta\hat{H}_0)^2 + (\Delta\hat{\Gamma})^2 - i\langle[\hat{H}_0,\hat{\Gamma}]\rangle$ is real valued and positive and is, indeed, bounded below by $\chi(t)^2 \geqslant (\Delta H_0 - \Delta\Gamma)^2 \geqslant 0$.

[1] D. Guéry-Odelin, A. Ruschhaupt, A. Kiely, E. Torrontegui, S. Martinez-Garaot, and J. G. Muga, Rev. Mod. Phys. **91**, 045001 (2019).
[2] J.-F. Schaff, X.-L. Song, P. Vignolo, and G. Labeyrie, Phys. Rev. A **82**, 033430 (2010).
[3] S. Martínez-Garaot, E. Torrontegui, X. Chen, M. Modugno, D. Guéry-Odelin, S.-Y. Tseng, and J. G. Muga, Phys. Rev. Lett. **111**, 213001 (2013).
[4] S. Amri, R. Corgier, D. Sugny, E. M. Rasel, N. Gaaloul, and E. Charron, Sci. Rep. **9**, 5346 (2019).
[5] F. Impens, R. Duboscq, and D. Guéry-Odelin, Phys. Rev. Lett. **124**, 250403 (2020).
[6] H. Zhou, Y. Ji, X. Nie, X. Yang, X. Chen, J. Bian, and X. Peng, Phys. Rev. Appl. **13**, 044059 (2020).
[7] B. B. Zhou, A. Baksic, H. Ribeiro, C. G. Yale, F. J. Heremans, P. C. Jerger, A. Auer, G. Burkard, A. A. Clerk, and D. D. Awschalom, Nat. Phys. **13**, 330 (2017)
[8] T. Wang, Z. Zhang, L. Xiang, Z. Jia, P. Duan, Z. Zong, Z. Sun, Z. Dong, J. Wu, Y. Yin, and G. Guo, Phys. Rev. Appl. **11**, 034030 (2019).
[9] F. M. D'Angelis, F. A. Pinheiro, D. Guéry-Odelin, S. Longhi, and F. Impens, Phys. Rev. Res. **2**, 033475 (2020).
[10] N. E. Palaiodimopoulos, I. Brouzos, F. K. Diakonos, and G. Theocharis, Phys. Rev. A **103**, 052409 (2021).
[11] T. Chasseur, L. S. Theis, Y. R. Sanders, D. J. Egger, and F. K. Wilhelm, Phys. Rev. A **91**, 043421 (2015).
[12] M. Demirplak and S. A. Rice, J. Phys. Chem. A **107**, 9937 (2003).
[13] M. Demirplak and S. A. Rice, J. Phys. Chem. B **109**, 6838 (2005).
[14] M. Demirplak and S. A. Rice, J. Chem. Phys. **129**, 154111 (2008).
[15] M. V. Berry, J. Phys. A: Math. Theor. **42**, 365303 (2009).
[16] H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[17] X. Chen and J. G. Muga, Phys. Rev. A **86**, 033405 (2012).
[18] Y.-Z. Lai, J.-Q. Liang, H. J. W. Müller-Kirsten, and J.-G. Zhou, Phys. Rev. A **53**, 3691 (1996); J. Phys. A **29**, 1773 (1996).
[19] A. Ruschhaupt, X. Chen, D. Alonso, and J. G. Muga, New J. Phys. **14**, 093040 (2012).
[20] F. Impens and D. Guéry-Odelin, Phys. Rev. A **96**, 043609 (2017).
[21] S. Deffner, C. Jarzynski, and A. del Campo, Phys. Rev. X **4**, 021013 (2014).
[22] S. Masuda and K. Nakamura, Phys. Rev. A **78**, 062108 (2008).
[23] S. Masuda and S. A. Rice, Phys. Rev. A **89**, 033621 (2014).
[24] K. Takahashi, Phys. Rev. A **89**, 042113 (2014).
[25] J.-J. Zhu and X. Chen, Phys. Rev. A **103**, 023307 (2021).
[26] I. Setiawan, B. Eka Gunara, S. Masuda, and K. Nakamura, Phys. Rev. A **96**, 052106 (2017).
[27] L. Mandelstam and I. G. Tamm, J. Phys. (USSR) **9**, 249 (1945).
[28] J. H. Eberly and L. P. S. Singh, Phys. Rev. D **7**, 359 (1973).
[29] J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65**, 1697 (1990).
[30] L. Vaidmann, Am. J. Phys. **60**, 182 (1992).
[31] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A **67**, 052109 (2003).
[32] L. B. Levitin and T. Toffoli, Phys. Rev. Lett. **103**, 160502 (2009).
[33] J. Batle, M. Casas, A. Plastino, and A. R. Plastino, Phys. Rev. A **72**, 032337 (2005).
[34] F. Fröwis, Phys. Rev. A **85**, 052127 (2012).
[35] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Phys. Rev. Lett. **110**, 050402 (2013).
[36] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Phys. Rev. Lett. **110**, 050403 (2013).
[37] Y. Ashida, Z. Gong, and M. Ueda, Adv. Phys. **69**, 249 (2020).
[38] S. Weidemann, M. Kremer, S. Longhi, and A. Szameit, Nat. Photonics **15**, 576 (2021).
[39] R. Arouca, E. C. Marino, C. Morais Smith, arXiv:2108.0747.
[40] F. Impens and D. Guéry-Odelin, Sci. Rep. **9**, 4048 (2019).
[41] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. **70**, 101 (1998).
[42] N. Moiseyev, Phys. Rep. **302**, 212 (1998).
[43] S. Martínez-Garaot, A. Ruschhaupt, J. Gillet, T. Busch, and J. G. Muga, Phys. Rev. A **92**, 043406 (2015).
[44] J. Xu, Y.-X. Du, and W. Huang, Phys. Rev. A **100**, 023848 (2019).

[45] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions: Basic Processes and Applications* (Wiley, New York, 1998), Chap. 3.
[46] H. Feshbach, Annals of Physics **5**, 357 (1958).
[47] G. L. Celardo and L. Kaplan, Phys. Rev. B **79**, 155108 (2009).
[48] J.-S. Li, J. Ruths, and S. J. Glaser, Nat. Commun. **8**, 446 (2017).
[49] N. V. Vitanov, A. A. Rangelov, B. W. Shore, and K. Bergmann, Rev. Mod. Phys. **89**, 015006 (2017).
[50] X.-K. Song, F. Meng, B.-J. Liu, D. Wang, L. Ye, and M.-H. Yung, Opt. Express **29**, 7998 (2021)