Taming the Time Evolution in Overdamped Systems: Shortcuts Elaborated from Fast-Forward and Time-Reversed Protocols

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Using a reverse-engineering approach on the time-distorted solution in a reference potential, we work out the external driving potential to be applied to a Brownian system in order to slow or accelerate the dynamics, or even to invert the arrow of time. By welding a direct and time-reversed evolution toward a well chosen common intermediate state, we analytically derive a smooth protocol to connect two arbitrary states in an arbitrarily short amount of time. Not only does the reverse-engineering approach proposed in this Letter contain the current—rather limited—catalog of explicit protocols, but it also provides a systematic strategy to build the connection between arbitrary states with a physically admissible driving. Optimization and further generalizations are also discussed.

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Shortcut to adiabaticity techniques originally aim at reaching adiabatic outcomes in a finite amount of time [1]. Adiabatic should be understood here in its quantum sense, synonymous with "sufficiently slowly driven" [2]. These methods have been extended to generate shortcuts between two states, regardless of the existence of an adiabatic connection between them [1]. While this field is rooted in quantum mechanics [3,4], related questions emerge in other domains, namely classical mechanics [5–11] and stochastic thermodynamics [11–18]. Yet, the question of transposing to statistical physics protocols originally developed in quantum mechanics is delicate. For instance, the so-called counterdiabatic protocol (also dubbed transitionless tracking) valid for any initial condition [19,20] can be directly transposed to overdamped dynamics [13]. However, the very same procedure yields nonconservative forcing, experimentally problematic to achieve, with underdamped systems [1,13]. Among the set of tools for accelerating the dynamics, other classes of solutions propose tailor-made protocols based on the specifics of the initial and final states, both in quantum mechanics [7,20–25] and in statistical physics [12,26–28]. More general optimal protocols in small thermodynamic systems have been obtained from a mapping to optimal transport, establishing an unexpected connection with cosmology but requiring a numerical resolution [29–33]. These protocols generically lead to discontinuous-in-time driving forces [26,30], which raises a delicate experimental challenge for implementation.

Thus, a central question is to work out shortcuts to transformations between arbitrary states that should involve nonsingular forces and be expressible in closed form. This is the question we solve in the present Letter, within the framework of the Fokker-Planck equation [34], which governs the evolution of the probability density $\rho(x,t)$ of Brownian objects, with x and t denoting position and time. To this end, we proceed in two steps. First, we show how to distort and control a reference dynamics, i.e., the time evolution taking place in a reference external potential $U_r(x,t)$, with a well chosen external potential $U(x,t)$. The realization of such driving is now achieved experimentally

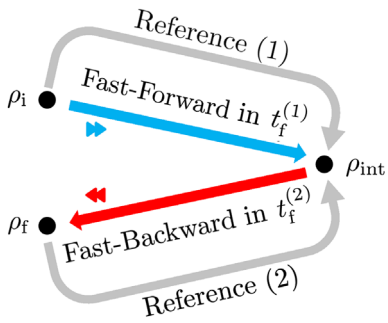

FIG. 1. Sketch of the welding strategy to connect two arbitrary states, ρ_i and ρ_f, through an auxiliary intermediate ρ_{int}. The reference processes are the direct relaxations toward ρ_{int} starting either from ρ_i or ρ_f. An acceleration of the first reference process connects ρ_i to ρ_{int} in a time $t_f^{(1)}$. The second reference process is time reversed and accelerated; it lasts $t_f^{(2)}$. Combining the two steps allows us to reach the target state ρ_f in a final time $t_f = t_f^{(1)} + t_f^{(2)}$, providing a fast shortcut from ρ_i to ρ_f. In general, ρ_{int} differs from both ρ_i and ρ_f. Otherwise, one of the steps (forward or backward) becomes unnecessary.

in a number of domains, from cold atom physics [35] to colloids [36], thanks to a proper steering of the optical potential applied to the system [16,17,37]. This will allow us to accelerate the reference dynamics [fast-forward (FF)] [21], or to decelerate it (slow-forward), and even to freeze the evolution. Interestingly, this also opens the way to reverse time's arrow and devise a backward drive at arbitrary speed. Second, we combine a fast-forward drive toward a suitably chosen intermediate state $\rho_{\rm int}(x)$ with a fast-backward (FB) protocol from $\rho_{\rm int}(x)$, to get a whole family of smooth shortcuts, as depicted in Fig. 1. In doing so, the system can be brought from a chosen state $\rho_i(x)$ to another chosen state $\rho_f(x)$, in an arbitrary time. Both states can be equilibrium or, more generally, out-of-equilibrium [38,39].

Reverse engineered potential.—The Brownian objects considered, such as colloids in dilute conditions, are immersed in a thermal bath at temperature T and submitted to an external force field stemming from a potential $U(x,t)$. The Fokker-Planck equation for the probability density $\rho(x,t)$ reads

$$\partial_x[\rho(x,t)\partial_x U(x,t)] = -\beta^{-1}\partial_x^2\rho(x,t) + \gamma\partial_t\rho(x,t), \quad (1)$$

where γ is the bath friction coefficient and $\beta = (k_B T)^{-1}$, with k_B the Boltzmann constant. A unique solution for $U(x,t)$ can be obtained by imposing a prescribed time evolution for the density $\rho(x,t)$, from an initial $\rho(x,0) = \rho_i(x)$ to a final $\rho(x,t_f) = \rho_f(x)$ state [40]. Once $\rho(x,t)$, together with the operating time t_f, has been chosen, Eq. (1) is nothing but a first order differential equation for the external force $-\partial_x U(x,t)$ that should be applied to the system to achieve the appropriate driving. Assuming that the density current vanishes at the boundaries of the system, $x = x_b$, we get

$$\partial_x U(x,t) = -\beta^{-1}\partial_x \ln\rho(x,t) + \frac{\gamma}{\rho(x,t)}\partial_t\int_{x_b}^{x} dx'\rho(x',t). \quad (2)$$

Unfortunately, except in a few cases, like that of a Gaussian distribution [12,26], shape preserving evolutions [41], or other very specific situations [33,41], Eq. (2) is impractical since it does not lead to an explicit closed-form potential $U(x,t)$ [42]. Thus, we seek an alternative route that provides a systematic approach to obtain admissible driving protocols in closed form.

From fast-forward to fast-backward.—The idea is to take advantage of the knowledge of a reference nontrivial dynamics for $\rho(x,t)$ to distort its time evolution by finding the appropriate driving potential. First, we show how a time contraction can be performed to force the system to reach equilibrium in a finite amount of time. Such a FF protocol realizes, in finite time, a process that takes an infinite amount of time when the system is unperturbed. Beyond FF, FB protocols—and, also, slow forward or backward ones—can also be engineered. Subsequently, we explain under which conditions the driving force remains continuous for all times, a key requirement for practical implementation.

Consider a reference process, i.e., a known solution $\rho_r(x,t)$ of the Fokker-Planck equation (1) in a confinement potential $U_r(x,t)$ over a time interval $0 < t < t_r$, which we write as a continuity equation

$$\partial_t\rho_r(x,t) = -\partial_x[\rho_r(x,t)v_r(x,t)], \quad (3a)$$

$$v_r(x,t) \equiv -\gamma^{-1}\partial_x[U_r(x,t) + \beta^{-1}\ln\rho_r(x,t)]. \quad (3b)$$

We define the desired prescribed density as

$$\rho(x,t) \equiv \rho_r(x,\Lambda(t)), \quad (4)$$

through a time manipulation of the reference process, embedded in the function $\Lambda(t)$. The prescribed and the reference evolution go through the same states, but displayed at a different frame rate and/or time ordering. Indeed, depending on the choice for $\Lambda(t)$, one can play "the movie" at FF speed ($\dot\Lambda > 1$), at slow motion ($0 < \dot\Lambda < 1$), pause it ($\dot\Lambda = 0$), or even rewind it: either at slow backward motion ($-1 < \dot\Lambda < 0$) or at FB speed ($\dot\Lambda < -1$). In the latter case, $\dot\Lambda < 0$, our procedure enables a new possibility: the design of a potential to invert time's arrow for an irreversible evolution obeying the Fokker-Planck equation.

Introducing Eqs. (3) and (4) into Eq. (2), and defining $\Delta U(x,t) = U(x,t) - U_r(x,\Lambda(t))$, which measures the departure from the reference potential, we get

$$\partial_x\Delta U(x,t) = [1-\dot\Lambda(t)]\gamma v_r(x,\Lambda(t)), \quad (5)$$

where the time dependence of the protocol is encapsulated in $\Lambda(t)$ and its derivative $\dot\Lambda(t)$. For simplicity, we restrict ourselves to $\Lambda(t)$ that are monotonic functions of time, either of FF [$\Lambda(0) = 0$ and $\Lambda(t_f) = t_r$] or of FB [$\Lambda(0) = t_r$ and $\Lambda(t_f) = 0$] type. The times t_r and t_f can be finite or infinite. Of particular relevance for experimental applications though is the shortcut of an infinite process (infinite t_r but finite t_f), e.g., for building irreversible nanoheat engines [28,43–50].

For the sake of simplicity, we consider hereafter a static reference potential $U_r(x)$. Such a potential is convenient; it makes it possible to analytically obtain $\rho_r(x,t)$ through an expansion in eigenmodes, as shown below [51]. Thus, with $t_r \to \infty$, the reference process is the relaxation to the equilibrium distribution for $U_r(x)$: $\rho_r \propto e^{-\beta U_r}$. In order to accelerate such an everlasting dynamics, we impose $\lim_{t\to t_f^-}\Lambda(t) = +\infty$. For our purposes, it is adequate to use the family of functions

$$\Lambda(t) = \tau f\left(\frac{t}{t_f}\right), \qquad f(z) = \frac{z^2}{(1-z)^\zeta}, \qquad \zeta > 0, \tag{6}$$

where τ is a characteristic time. The divergence of $\Lambda(t)$ at $t = t_f$ implies that $\lim_{t\to t_f^-} \dot{\Lambda}(t) = +\infty$, which suggests that $\Delta U(x,t)$ may diverge at the final time. However, the velocity field $v_r(x, \Lambda(t))$ vanishes at t_f, and thus, the behavior of $\Delta U(x, t_f^-)$ needs to be elucidated.

Regularity of the driving potential.—The family defined in Eq. (6) has the property $\dot{\Lambda}(0) = 0$. This guarantees the continuity of the force at the initial time, as implied by Eq. (5). Now, we show, on general grounds, that the force field remains continuous even at t_f, provided the reference potential $U_r(x)$ is confining—in the sense that there exists a well-defined equilibrium density. The proof uses the expansion in eigenmodes for the solution of the reference process and a mapping to a quantum problem. Introducing $\psi(x,t) = \rho_r(x,t)e^{\beta U_r(x)/2}$ leads to the time-dependent Schrödinger equation $\partial_t \psi = -H\psi$ with Hamiltonian [52],

$$H = -\frac{1}{\gamma\beta}\partial_x^2 + \frac{1}{2\gamma}\left[\frac{\beta U_r'(x)^2}{2} - U_r''(x)\right]. \tag{7}$$

The smallest eigenvalue, associated with the equilibrium distribution, is zero: $He^{-\beta U_r(x)/2} = 0$. The other eigenvalues of H are positive, $0 < \lambda_1 < \lambda_2 < \cdots$, implying that the corresponding modes decay exponentially in time. Specifically, $H\varphi_n(x) = \lambda_n \varphi_n(x)$, where $\varphi_n(x)$ is the eigenfunction associated to the nth eigenvalue [53].

The expansion of the reference process in the eigenbasis reads

$$\rho_r(x,t) = \frac{e^{-\beta U_r(x)}}{Z_r} + e^{-\beta U_r(x)/2}\sum_{n=1}^{\infty} c_n \varphi_n(x) e^{-\lambda_n t}, \tag{8}$$

where $Z_r = \int_{\mathcal{D}} dx e^{-\beta U_r(x)}$ is the partition function and $c_n = \langle \varphi_n | \psi(t=0)\rangle = \int_{\mathcal{D}} dx \varphi_n(x)\psi(x,0)$. We are interested in the limit $t \to t_f^-$, for which $\Lambda(t) \to +\infty$. From Eq. (8), we deduce

$$\rho_r(x, \Lambda(t)) \sim \frac{e^{-\beta U_r(x)}}{Z_r}(1 + \tilde{c}_1 \chi_1(x) e^{-\lambda_1 \Lambda(t)}), \tag{9}$$

where $\tilde{c}_n \equiv Z_r c_n$ and $\chi_n(x) \equiv e^{\beta U_r(x)/2}\varphi_n(x)$. As a result, to the lowest order in $e^{-\lambda_1\Lambda(t)}$, we get

$$\partial_x \ln \rho_r(x, \Lambda(t)) \sim -\beta \partial_x U_r(x) + \tilde{c}_1 \partial_x \chi_1(x) e^{-\lambda_1 \Lambda(t)}, \tag{10}$$

which, combined with Eq. (5), gives

$$\partial_x \Delta U(x,t) \sim -\beta^{-1}\frac{\tilde{c}_1}{\lambda_1}\frac{de^{-\lambda_1\Lambda(t)}}{dt}\partial_x \chi_1(x). \tag{11}$$

There is a vast set of diverging functions $\Lambda(t)$ including the family (6) that forces the cancellation of the right hand side of Eq. (11) when t approaches t_f [42], a requirement that implies that the driving force remains continuous at the final time. In our proof, the existence of a well-defined partition function Z_r is important. This is not the case for a free expansion, $U_r(x) = 0$, in an infinite or semi-infinite space [42]. However, the argument remains valid for a reference free diffusing system within a finite box, as shown below.

To sum up, Eq. (5) provides the smooth potential needed to tune, at will, the reference process according to the time mapping function $\Lambda(t)$. Despite the limited number of analytically solvable reference processes [42], we can always rely on an expansion as in Eq. (8), but with a certain cutoff—examples are provided below. In doing so, the tailored process does not strictly reach the target state, but its distance thereto—e.g., with the L^2 norm—can be made as small as desired [42,54].

Diffusion in a box.—We illustrate the time manipulation procedure with the FF transformation of the free diffusion for a dilute gas within a finite box, $x \in [0, L]$. The gas is initially at equilibrium in the presence of a constant and homogeneous force, $\rho_r(x,0) \propto e^{\beta F x}$, thus, with a sedimentation profile. In the reference process, at time $t = 0$, the force is suddenly removed, and it takes an infinite time to reach the stationary homogeneous state. The time scale of the relaxation is characterized by $\lambda_1^{-1} = \gamma\beta L^2/\pi^2$.

We have chosen $\Lambda(t)$ from Eq. (6) with $\zeta = 1$, $\tau = \lambda_1^{-1}$, and $t_f = \tau/10$. The results for the FF transformation are displayed in Fig. 2, for the density of the gas and the force $-\partial_x U(x,t)$. Apart from a short time window where the forces are negative because of continuity, the forces required to accelerate homogenization are positive, pushing the system in the direction opposite to that of the initial force. As highlighted above, the protocol is smooth in time, including the initial and final times. In general, faster accelerations have a higher cost. Both the magnitude of the required force and the excess irreversible work increase as t_f is decreased [42].

Connecting arbitrary states.—We come back to the welding idea conveyed in Fig. 1. For the sake of concreteness, the initial and final densities are the equilibrium distributions corresponding, respectively, to the initial and final potential $U_i(x) \propto x^4$ and $U_f(x) \propto x^6$. None of these two potentials provides a convenient reference potential: the associated reference dynamics cannot be analytically solved and the FF idea results inoperative. Here, the welding method offers a solution. Indeed, choosing a Gaussian intermediate state ($\log \rho_{\rm int} \propto -x^2$) is free of the above difficulties, and solvable. Thus, we take $\beta U_r(x) = x^2/(2\sigma_x^2)$, where σ_x^2 stands for the variance of $\rho_{\rm int}$ [42]. As previously, we accelerate the forward (FF) and backward (FB) processes by a factor of 10, $t_f = t_f^{(1)} = t_f^{(2)} = \tau/10$,

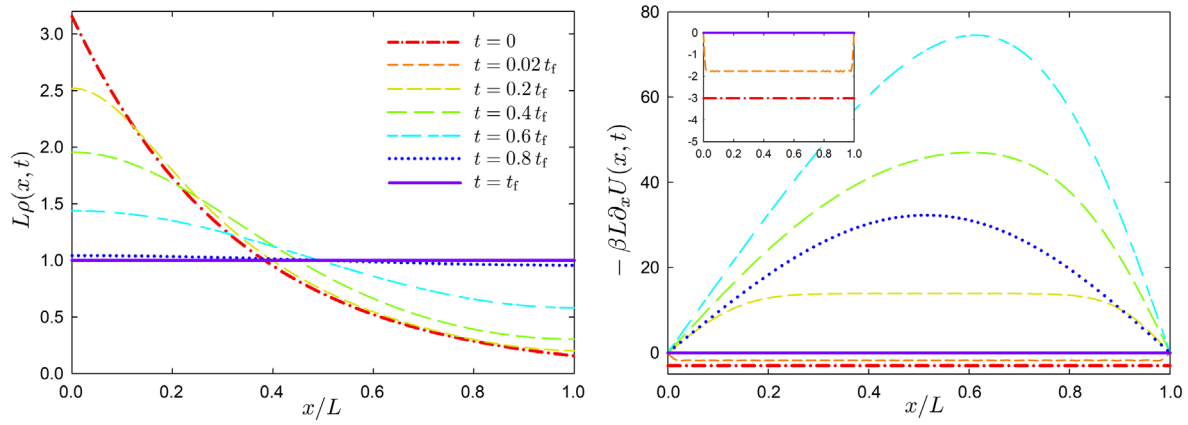

FIG. 2. Accelerated free diffusion in a finite-size box of size L. Starting from the exponential profile $\rho_r(x,0) \propto e^{\beta F x}$, the subsequent reference solution ρ_r corresponds to a force-free evolution, $U_r(x) = 0$. This leads to strict equilibrium in an infinite time, featuring a characteristic (diffusive) time τ. This reference dynamics is accelerated such that at $t_f = \tau/10$, the density profile is strictly that at equilibrium [flat, i.e., $\rho(x,t_f) = 1/L$]. The time evolution of the density is displayed in the left panel, whereas the right panel shows the force required to drive such an accelerated transformation. The inset shows a zoomed region for negative forces. Here, $\beta F L = -3$ (as conventional for a gravitational field, we take $F < 0$). For the numerical evaluation, the cutoff in the expansion (8) is $n_{\rm cut} = 70$ [42].

with $\tau = \lambda_1^{-1} = \gamma\beta\sigma_x^2$. For both the FF and FB steps, the Λ function is taken to be the same, with $\zeta = 1$ [55]. The results for the welded FF and FB protocols are displayed in Fig. 3. In the first step, the tails of the density have to be pushed away from the center and the confinement needs to be strengthened in the central region to reach faster $\rho_{\rm int}$; hence, the N-shape force. During the second stage, the requirements are opposite, leading to an inverted N shape. The force protocol is continuous for all times, including the initial and final times for both steps of the welding strategy.

We have focused on a specific functional shape for the time manipulation, Eq. (6), in order to ensure the continuity properties that we desire for the protocol. One may consider optimization problems, such as finding the time manipulation that minimizes some relevant observable—like the excess irreversible work [29–33]. Remarkably, it is possible to show that such an optimization over all FF protocols [attached to a given U_r but with an arbitrary $\Lambda(t)$] lead to processes delivering excess work at constant rate, as happens with the full optimization [42]. Besides, when it comes to connecting two Gaussian states, the optimal FF protocol coincides with the full, unconstrained, optimum—and the minimum restricted to the specific $\Lambda(t)$ family in Eq. (6) lies only 4% above it, for a ten-fold expansion of the state [42].

To sum up, we have developed a reverse-engineering technique in order to manipulate, at will, the time evolution of a reference process. This provides us with the external potential required to reach a target distribution in a desired time. Interestingly, not only does the framework allow for

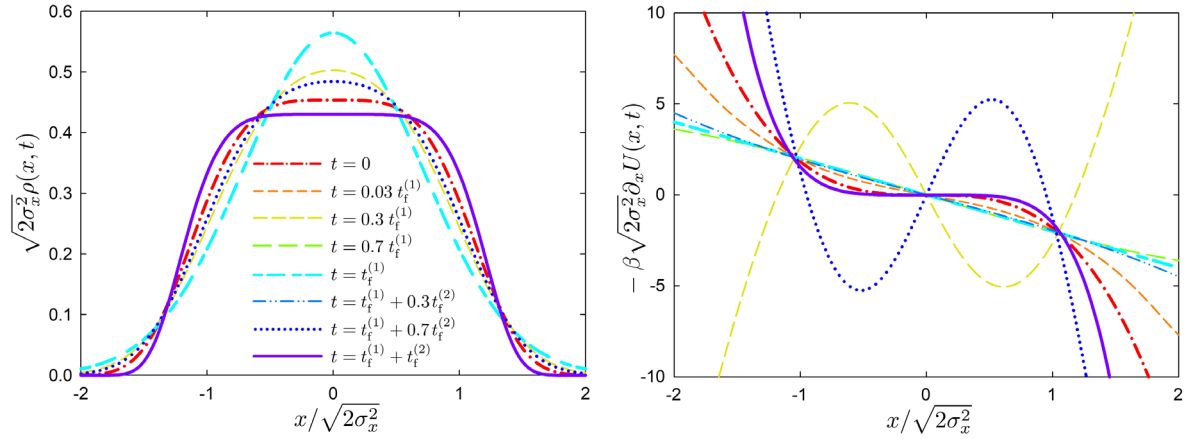

FIG. 3. Illustration of an operational welding connection. Left: time evolution of the distribution. Right: driving force to be applied. The intermediate state is Gaussian with variance σ_x^2, associated to the reference potential $U_r(x) = x^2/(2\sigma_x^2)$. The initial (final) state is the thermal Boltzmann distribution in the potential $U_i(x) \propto x^4$ [$U_f(x) \propto x^6$]. The initial, final, and reference densities are chosen to have the same variance. The transformation is performed in a finite time $t_f = t_f^{(1)} + t_f^{(2)}$ with $t_f^{(1)} = t_f^{(2)} = \tau/10$; $n_{\rm cut} = 70$ [42].

the acceleration of forward processes but also for the inversion of time's arrow. Taking these time manipulated reference processes as building blocks, we have put forward a neat welding procedure to connect two arbitrary states in an arbitrarily small finite time. The method relies on a nonlinear time mapping of two relaxation processes in the same harmonic potential; it produces, by construction, a smooth driving potential, with continuous force field, for all times. Finally, our procedure can be generalized to higher dimensional problems [42,56]. Some possible venues for developments lie in the thermalization processes to build irreversible nanoheat engines [12,28,43–47,49,50] or the genetic control of an evolving population [57,58].

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[40] For the moment, we focus our attention on single-step connections. The concept of the intermediate state will return when designing the welding strategy.

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[42] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.127.190605 for a discussion of shape-preserving dynamics (which includes a number of previously proposed explicit protocols) along with other simple proposals, the analysis of the continuity property of the steering potential, the dependence of the force intensity on the duration of the process, the study of optimal properties, the spectral decomposition backing up the welding procedure, the convergence of the resulting expansion, and the generalization to higher dimensions. A Mathematica notebook solving the welding problem is also provided.

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[53] The eigenfunctions are mutually orthogonal and normalized, $\langle\varphi_m|\varphi_n\rangle = \int_{\mathcal{D}} dx \varphi_m(x)\varphi_n(x) = \delta_{mn}$, where $\mathcal{D}$ denotes the integration domain.

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