On the existence and optimisation of finite-time adiabatic processes

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The concept of adiabaticity is ubiquitous in physics, and often associated to long time operations. We show here how this notion can be rigorously extended to finite-time processes in the context of stochastic thermodynamics. The case of a trapped Brownian particle in the overdamped regime is explicitly worked out. Optimal features are subsequently addressed. We prove that finding (i) the minimum operating time of any protocol connecting prescribed equilibrium states and (ii) the optimal final temperature for a prescribed protocol duration are intimately related, and ruled by a unique control function. Such fast adiabatic transformations are particularly relevant for the optimal design of nanoengines.

Adiabatic processes are a cornerstone in the thermodynamics of macroscopic systems. Therein, energy is solely exchanged as work–there is no heat. If, in addition, the system always sweeps equilibrium states, that is, the process is reversible, there is no entropy change. These processes are essential to build the Carnot heat engine, which consists of two reversible isothermal and two reversible adiabatic branches [1][2].

Stochastic thermodynamics deals with mesoscopic systems, the smallness of which entails that fluctuations are relevant [3, 4]. It is therefore meaningless to imagine a Brownian particle that would be thermally isolated from its environment for each of its trajectories: over them, both work and heat contribute to the energy change [5, 6]. However, one can think of processes in which the average heat vanishes and thus the average work gives the average energy increment. This is the concept of adiabatic process in stochastic thermodynamics, which is essential–for example–to build a finite-time version of the Carnot engine. In this context, many recent works deal with cyclic heat engines that are said to incorporate adiabatic branches [6–12]. Notwithstanding, and to the best of our knowledge, finite-time adiabatic processes have not been rigorously defined nor characterised yet.

There have been two approaches to the building of adiabatic processes in the literature. Several works, mainly in the overdamped limit, have considered *instantaneous* processes in which the position distribution does not change [7, 9, 10, 12]. Since the configurational contributions to the heat–and to the entropy change–vanish for such processes, they have been termed *adiabatic*. Nevertheless, as pointed out in Refs. [7, 13], there is a contribution to the heat–and to the entropy change–coming from the velocity degree of freedom, because the temperature varies in time. Thus, these instantaneous processes are not actually adiabatic.

Other works have considered *reversible*, quasi-static, adiabatic processes [6, 8, 11]. This approach has been carried out within the underdamped description, which incorporates the velocity degree of freedom to the picture. Specifically, it has been shown how *reversible adiabatic* processes are built [8], for which –consistently–there is no entropy increment [11]. Very recently, a Brownian Carnot engine consisting of two irreversible isothermal branches and two reversible adiabatic ones has been investigated [6]. These approaches are operational for quasi-static, infinite time protocols.

Hereafter, we answer two physically relevant questions. First, we show that *finite-time adiabatic* processes can be rigorously constructed within the overdamped formalism, where inertial effects are not relevant [14]. This requires time control of both the confinement strength and the bath temperature, which is experimentally relevant, e.g. for micron-size colloids in a suspending fluid and manipulated by laser tweezers [15, 16][17].

Second, we address two different kinds of optimisation of these *finite-time adiabatic* processes. We minimise the running time for protocols connecting two equilibrium states with given values of the stiffness and the temperature. We then optimise the final temperature for given connection time and final value of the stiffness. Indeed, an adiabatic transformation cannot be isothermal and depending on the target state, the final temperature of the system may be large. It is thus of interest to minimise the mismatch with the initial temperature. Interestingly, these two optimisation problems turn out to be closely related: a unique function provides the solution to both of them.

Our system is a Brownian particle immersed in a heat bath at temperature T and trapped in a harmonic potential with stiffness k. We denote the variance of the particle position by $\langle x^2\rangle$. At any time t, the state of the system is characterised by the state-point $(k, \langle x^2\rangle, T)$. In dimensionless variables, the time evolution of the variance y in the overdamped regime reads [18]

$$\dot{y} = -2\kappa y + 2\theta. \tag{1}$$

Stiffness and temperature have been made dimensionless with some arbitrary values k_0 and T_0, whereas the time unit is λ/k_0 and λ is the viscous drag. Finally, the unit of the variance y_0 has been chosen as the equilibrium value $k_B T_0/k_0$. Equilibrium states thus correspond to the surface $\kappa y = \theta$ in the space (κ, y, θ).

Consider the energetics of this system at the average level. Energy has two contributions: a harmonic one, stemming from the trap, and a kinetic one. The latter always has the equilibrium value $\theta/2$ in the overdamped limit. Thus, the average energy and its equilibrium value are $\mathcal{E} = (\kappa y + \theta)/2$ and $\mathcal{E}_{\rm eq} = \theta$, respectively. Infinitesimal work and heat are $đW = y\, d\kappa/2$ and $đQ = (\kappa\, dy + d\theta)/2$, so that the first law $d\mathcal{E} = đQ + đW$ holds [3]. The energy unit for non-dimensionalisation is $k_B T_0$.

Now we turn our attention to the building of adiabatic processes, in which there is no heat transfer–in average– at all times. The condition $đQ = 0$ entails that

$$\kappa\, dy + d\theta = 0. \tag{2}$$

Temperature becomes a function of time $\theta(t)$ that goes from $\theta_{\rm i}$ to $\theta_{\rm f}$ [19]. In this paper, we restrict ourselves to processes connecting two equilibrium states $(\kappa_{\rm i}, y_{\rm i}, \theta_{\rm i})$ and $(\kappa_{\rm f}, y_{\rm f}, \theta_{\rm f})$, and then $\kappa_{\rm i} y_{\rm i} = \theta_{\rm i}$, $\kappa_{\rm f} y_{\rm f} = \theta_{\rm f}$.

The average energetics is simple. The change in energy is given by the change in temperature, $\mathcal{E}_{\rm i} = \theta_{\rm i}$, $\mathcal{E}_{\rm f} = \theta_{\rm f}$, and $\Delta\mathcal{E} \equiv \mathcal{E}_{\rm f} - \mathcal{E}_{\rm i} = \theta_{\rm f} - \theta_{\rm i}$. Since there is no heat exchange, the total work coincides with the energy change, $W_{\rm i}^{\rm f} = \Delta\mathcal{E} = \theta_{\rm f} - \theta_{\rm i}$. These equalities apply for any adiabatic process, regardless of its duration, being valid for both quasi-static and finite-time processes. Without loss of generality, we choose in the following units such that $\kappa_{\rm i} = \theta_{\rm i} = 1$ and then $y_{\rm i} = 1$ [20].

In quasi-static adiabatic processes, $\kappa(t)$ and $\theta(t)$ are tuned in an infinitely slow way so that the path followed by the system is the equilibrium curve at all times. Using the adiabaticity condition (2) and the equilibrium relation $\kappa y = \theta$, one gets

$$y_\infty(t) = [\theta_\infty(t)]^{-1} = [\kappa_\infty(t)]^{-1/2}, \tag{3}$$

in which the subindex ∞ refers to quasi-static process [6, 8, 11].

For finite-time adiabatic processes, Eqs. (1) and (2) imply the inequality

$$\frac{d}{dt}(y\theta) = \frac{1}{2}\dot{y}^2 \geq 0. \tag{4}$$

Therefore, $y(t)\theta(t)$ monotonically increases and $y(t)\theta(t) \geq 1$. Two arbitrary states cannot be connected with an adiabatic transformation. For the final time, Eq. (4) yields $\theta_{\rm f}\, y_{\rm f} \geq 1$ or, equivalently,

$$\theta_{\rm f} \geq \theta_{\rm f}^{\infty} \equiv \sqrt{\kappa_{\rm f}}. \tag{5}$$

The equality corresponds to the quasi-static case.

We show hereafter that there exists a minimum time to carry out an attainable adiabatic process. The question is to find the optimal driving $\kappa(t)$ and $\theta(t)$ reaching the target state, where equilibrium demands a specific variance at final time $y_{\rm f} = \theta_{\rm f}/\kappa_{\rm f}$.

Integrating the left-hand-side (lhs) of Eq. (4) with respect to time from 0 to $t_{\rm f}$ and introducing a normalised time $\tau \equiv t/t_{\rm f}$ on its right-hand-side (rhs) yields

$$\Delta(y\theta) = \frac{1}{2t_{\rm f}} J[y], \text{ with } J[y] \equiv \int_0^1 d\tau \left(\frac{dy}{d\tau}\right)^2, \tag{6}$$

where Δ refers to the differences between final and initial states. For a given target state, the lhs of Eq. (6) has a fixed value, $\Delta(y\theta) = \Delta(\theta^2/\kappa)$. Thus, since the functional $J[y]$ reaches a minimum value for a certain profile $\tilde{y}(\tau)$ that does not depend on $t_{\rm f}$, the connecting time also reaches a minimum value $\tilde{t}_{\rm f}$ for that optimal profile.

It is worth discussing the continuity of $(\kappa(t), y(t), \theta(t))$ as functions of time. Eqs. (1) and (2) involve $\dot{y}$ and $\dot{\theta}$, so both $y(t)$ and $\theta(t)$ must be continuous throughout the whole time interval $[0, t_{\rm f}]$, including its endpoints. There are no such constraints on the stiffness, for which piecewise continuity of $\kappa(t)$ suffices–finite jumps cannot be ruled out [21–24].

The functional $J[y]$ is minimised by the linear profile

$$\tilde{y}(\tau) = 1 + \tau\, \Delta y = 1 + \tau\, \Delta(\theta/\kappa). \tag{7}$$

Substitution of Eq. (7) into (6) gives

$$\tilde{t}_{\rm f} = \frac{(\Delta y)^2}{2\,\Delta(y\theta)} = \frac{[\Delta(\theta/\kappa)]^2}{2\,\Delta(\theta^2/\kappa)}, \tag{8}$$

which is the minimum time for the process: two equilibrium states cannot be connected in a time $t_{\rm f} < \tilde{t}_{\rm f}$. This result can also be obtained by solving a variational problem with constraints [18]. The positivity of $\tilde{t}_{\rm f}$ is guaranteed by Eq. (5).

The corresponding evolution of temperature is obtained by integrating Eq. (4) from 0 to t, with the result

$$\tilde{\theta}(\tau) = \frac{1 + \tau\, \Delta\left(\theta^2/\kappa\right)}{1 + \tau\, \Delta(\theta/\kappa)}, \tag{9}$$

where use has been made of Eq. (8). Following our discussion on continuity, Eqs. (7) and (9) are valid in the whole time interval $t \in [0, \tilde{t}_{\rm f}]$ or $0 \leq \tau \leq 1$. Both $\tilde{y}$ and $\tilde{\theta}$ are monotonic functions of time, the sign of their derivatives being those of Δy and $\Delta\theta$, respectively.

The optimal stiffness for $0 < \tau < 1$ follows from the adiabaticity condition (2), which gives after some simple algebra

$$\tilde{\kappa}(\tau) = \frac{C}{\tilde{y}(\tau)^2}, \quad C = \frac{\Delta\theta}{\Delta(1/y)} = \frac{\Delta\theta}{\Delta(\kappa/\theta)}. \tag{10}$$

This expression has a well-defined sign, since Eq. (2) entails that $\kappa = -\dot{\theta}/\dot{y}$ and both y and θ are monotonic [25].

We start by analysing a *deconfining* process ($\kappa_\mathrm{f} < 1$). Adiabaticity implies that $\theta_\mathrm{f} \geq \theta_\mathrm{f}^\infty = \sqrt{\kappa_\mathrm{f}} > \kappa_\mathrm{f}$. Thus, $y_\mathrm{f} = \theta_\mathrm{f}/\kappa_\mathrm{f} > 1$ and $\Delta y > 0$: the system always expands in the optimal protocol. For fixed κ_f, $\tilde{t}_\mathrm{f}$ is a non-monotonic function of θ_f: $\tilde{t}_\mathrm{f}$ decreases from infinity for the quasi-static value $\theta_\mathrm{f} = \theta_\mathrm{f}^\infty$ to $t^d_\mathrm{min} = (2\kappa_\mathrm{f})^{-1} - 1/2$ for $\theta_\mathrm{f} = 1$, and increases therefrom to $t_\mathrm{f}^{(1)} = (2\kappa_\mathrm{f})^{-1}$ when $\theta_\mathrm{f} \to +\infty$. Consider now the *confining* case ($\kappa_\mathrm{f} > 1$). The adiabaticity condition (5) implies that $\theta_\mathrm{f} > 1$. Therefore, the system always heats in the optimal process, $\Delta\theta > 0$, but it may compress, $\Delta y < 0$, or decompress, $\Delta y > 0$ [26]. Again, for fixed κ_f, the minimum time $\tilde{t}_\mathrm{f}$ shows a non-monotonic behaviour as a function of θ_f: $\tilde{t}_\mathrm{f}$ decreases from infinity at $\theta_\mathrm{f} = \theta_\mathrm{f}^\infty$ to $t^c_\mathrm{min} = 0$ at $\theta_\mathrm{f} = \kappa_\mathrm{f}$ ($\Delta y = 0$), and increases therefrom to $t_\mathrm{f}^{(1)}$ for $\theta_\mathrm{f} \to +\infty$.

These behaviours are illustrated in Fig. 1. On the top (bottom) panel, we plot $\tilde{t}_\mathrm{f}$ as a function of θ_f for the deconfining (confining) case, specifically $\kappa_\mathrm{f} = 0.25$ ($\kappa_\mathrm{f} = 5$). The horizontal dashed red line marks the minimum time $t^{d\text{ or }c}_\mathrm{min}$, the horizontal blue dashed line the asymptotic value $t_\mathrm{f}^{(1)}$, and the dotted vertical asymptote the minimum temperature for an adiabatic process θ_f^∞.

The main difference emerging in the confining case is the vanishing of the minimum value of $\tilde{t}_\mathrm{f}$ as a function of θ_f, $t^c_\mathrm{min} = 0$. Instantaneous adiabatic process are *only possible in the confining case*. Therein, the system is neither compressed nor decompressed, $\Delta y = 0$. The "price" for building such a process is an instantaneous round trip to infinite stiffness, $\kappa_\mathrm{i} \to \infty \to \kappa_\mathrm{f}$, because C diverges when $\Delta y = 0$. For both confinement and deconfinement, the optimal stiffness $\tilde{\kappa}(t)$ is positive (negative) for final temperatures θ_f to the left (right) of the one at which the global minimum time $t^{d\text{ or }c}_\mathrm{min}$ is attained. Specifically, the sign of $\tilde{\kappa}(t)$ is that of the constant C. For deconfining (confining), Δy ($\Delta\theta$) is positive but $\Delta\theta$ (Δy) changes sign at the minimum. A density plot of $\log_{10} C$ in the $(\kappa_\mathrm{f}, \theta_\mathrm{f})$ plane is depicted in Fig. 2.

There is another–in principle different–physically relevant optimisation problem for adiabatic processes. Starting from a given initial equilibrium state, consider all possible time evolutions of the stiffness $\kappa(t)$ that last a given time t_f and end up in a final equilibrium state with a given value of the stiffness κ_f. Hence the question: what is the extremal value of the final temperature θ_f?

Above, we have shown the existence of a minimum connection time for a given final value of the temperature. By fixing the connection time, we are preventing the system from reaching final temperatures that require longer optimal times. Then, we have to seek solutions of the equation $\tilde{t}_\mathrm{f}(\tilde{\theta}_\mathrm{f}, \kappa_\mathrm{f}) = t_\mathrm{f}$. Equivalently, by inverting Eq. (8), one gets the quadratic equation

$$\left(1 - \frac{1}{2\kappa_\mathrm{f} t_\mathrm{f}}\right)\tilde{\theta}_\mathrm{f}^2 + \frac{1}{t_\mathrm{f}}\tilde{\theta}_\mathrm{f} - \kappa_\mathrm{f}\left(1 + \frac{1}{2t_\mathrm{f}}\right) = 0, \qquad (11)$$

which, again, also follows from a variational approach [18]. Its solution(s) $\tilde{\theta}_\mathrm{f}$ can be understood as the optimal value(s) of the temperature for given t_f, as explained below by taking a fresh look at Fig. 1. On both panels, confinement and deconfinement, the situation is similar and thus we discuss them together in the following.

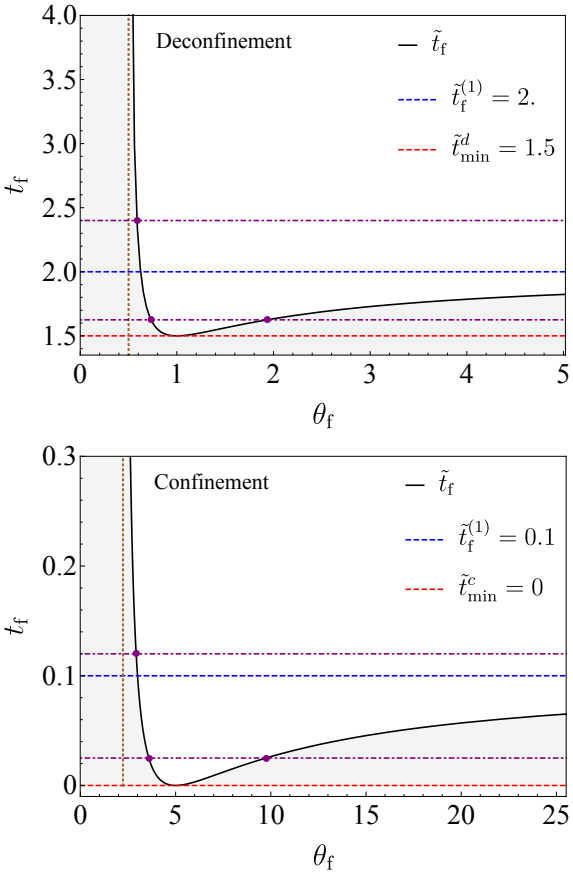

FIG. 1. Minimum connecting time as a function of the target temperature. The top (bottom) panel shows the deconfining (confining) case with $\kappa_\mathrm{f} = 0.25$ ($\kappa_\mathrm{f} = 5$). The greyed area corresponds to the forbidden region. The minimum time $\tilde{t}_\mathrm{f}$ is non-monotonic: on the top (bottom) panel, it decreases from infinity at the quasi-static limit $\theta_\mathrm{f} \to \theta_\mathrm{f}^\infty = \sqrt{\kappa_\mathrm{f}}$ to its minimum value t^d_min (t^c_min) at $\theta_\mathrm{f} = 1$ ($\theta_\mathrm{f} = \kappa_\mathrm{f}$) for deconfinement (confinement). Therefrom, it increases to $t_\mathrm{f}^{(1)} = (2\kappa_\mathrm{f})^{-1}$ in the limit as $\theta_\mathrm{f} \to \infty$ on both panels. The main difference between confinement and deconfinement is that $t^d_\mathrm{min} \neq 0$ whereas $t^c_\mathrm{min} = 0$. This entails the impossibility of engineering an instantaneous adiabatic process in deconfinement. The optimisation problem of the temperature for fixed connection time t_f, which is controlled by the same function $\tilde{t}_\mathrm{f}(\theta_\mathrm{f}, \kappa_\mathrm{f})$, is illustrated with the help of the horizontal purple dot-dashed lines. Specifically, the two lines on each panel correspond to different connecting times t_f, one above and one below $t_\mathrm{f}^{(1)}$. The purple points mark their intersections with $\tilde{t}_\mathrm{f}$, which are the optimal temperature(s) for that value of t_f.

For connecting times t_f longer than $t_\mathrm{f}^{(1)}$, temperatures below the only one verifying $\tilde{t}_\mathrm{f}(\tilde{\theta}_\mathrm{f}, \kappa_\mathrm{f}) = t_\mathrm{f}$ are inaccessi-

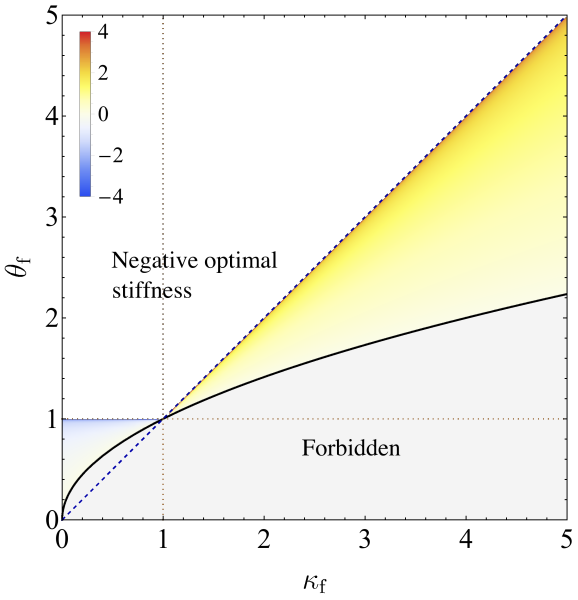

FIG. 2. Density plot of $\log_{10} C$ in the (κ_f, θ_f) plane, for those regions inside which $C > 0$. The solid black curve $\theta_f = \theta_f^{\infty} = \sqrt{\kappa_f}$ corresponds to the quasi-static limit, for which the minimum connecting time $\tilde{t}_f$ diverges. Thus, it demarcates the region (greyed, labelled "Forbidden") in which adiabatic processes are not possible. The dashed blue line stands for $\theta_f = \kappa_f$, at which $\Delta y = 0$–the variance does not change. It is only accessible for confining, $\kappa_f > 1$, and $\tilde{t}_f = t^c_{\min} = 0$ over it. In the empty region delimited by the lines $\theta_f = 1$ ($\kappa_f < 1$) and $\theta_f = \kappa_f$ ($\kappa_f > 1$), it is $C < 0$ (labelled "Negative optimal stiffness").

ble. Then, $\tilde{\theta}_f$ is the minimum temperature that can be attained with an adiabatic process of duration t_f. Such a situation is illustrated with the horizontal dot-dashed line above $t_f^{(1)}$, over which $\tilde{\theta}_f$ is marked–the positive solution of Eq. (11). For $t^{d \text{ or } c}_{\min} \leq t_f \leq t_f^{(1)}$, there are two temperatures verifying $\tilde{t}_f(\tilde{\theta}_f, \kappa_f) = t_f$. Neither temperatures below the smaller nor temperatures above the larger can be achieved, because they demand a longer t_f. Similarly, we illustrate this case with the horizontal dot-dashed line below $t_f^{(1)}$, over which the minimum and maximum temperatures are marked–the two solutions of Eq. (11) are positive. Thus, there appears an interval of reachable temperatures in this region of short connecting times.

Now we highlight some physical differences between deconfinement and confinement. For deconfining, the minimum temperature for fixed t_f provides an upper bound for $|\Delta\theta|$ in the cooling region $\theta_f < 1$. As t_f decreases, the minimum temperature increases and, for $t_f < t_f^{(1)}$, there also appears a maximum temperature in the heating region $\theta_f > 1$. The two of them converge to $\theta_f = 1$ as t_f approaches its global minimum $t^d_{\min} \neq 0$. For confining, the minimum temperature represents a strictly positive lower bound for $\Delta\theta$. This is physically relevant: it limits the range of temperatures swept in the time evolution–recall that $\tilde{\theta}(t)$ is monotonic. As t_f decreases, again the minimum temperature increases and a maximum temperature emerges for $t_f < t_f^{(1)}$, with them coalescing at $\theta_f \to \kappa_f$ for $t_f \to t^c_{\min} = 0$–the instantaneous adiabatic process for which the variance remains unchanged.

To conclude, we have rigorously characterised finite-time adiabatic processes, which have been furthermore optimised. For the sake of concreteness, this has been done in a model system that constitutes a benchmark for stochastic thermodynamics: a Brownian particle trapped in a harmonic well of stiffness κ and immersed in a fluid at temperature θ, which is characterised by its position variance y. Our driving functions are κ and θ, the time dependence of which is controlled. The optimisation problems we have addressed are relevant for experiments. For example, the minimum time imposes limits on the engineering of irreversible heat engines and the range of reachable temperatures in the laboratory is limited.

Our work paves the way for the construction of irreversible heat engines with *actual* finite-time adiabatic branches. Our results entail the impossibility of building a Carnot-like engine with two isotherms and two instantaneous adiabatic processes: the *instantaneous cooling adiabatic* branch does not exist. On a physical basis: the adiabatic condition $\kappa dy + d\theta = 0$ implies that an instantaneous change of temperature $\Delta\theta$ needs $\kappa \to \infty$ and $\Delta y \to 0$. In addition, the system must heat, $\Delta\theta \geq 0$, because $y\theta$ is a non-decreasing function of time.

Although our analysis has been performed in the overdamped limit, the existence of a minimum time for carrying out an adiabatic process is expected to be still valid in the underdamped description–even strengthened. From a physical point of view, it is reasonable to expect that the vanishing of $t^c_{\min}$ is linked to the vanishing of the relaxation time of the velocity degree of freedom in the overdamped limit. Thus, the corresponding $t^c_{\min}$ in the underdamped description would be non-zero and an instantaneous adiabatic process would be forbidden, for both confinement and deconfinement.

Among the perspectives opened by our results, it is interesting to address the stability of the optimal solutions found here with respect to small perturbations in the stiffness of the trap [27]. This is relevant for experiments, as a consequence of fluctuations and the unavoidable non-perfect implementation of the optimal protocols. On another note, our classical approach may be used to guide the extension of the concept of finite-time adiabaticity to open quantum systems [28–30].

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[18] See the Supplemental Material at [URL will be inserted by publisher] for further details on the variance equation, and for alternative variational calculations in the two extremal problems.

[19] Initial and final values of the physical quantities are denoted by subindexes i and f, respectively.

[20] With other choice for the units, all expressions in the paper remain valid with the substitutions $\kappa_{\rm f} \to \kappa_{\rm f}/\kappa_{\rm i}$, $\theta_{\rm f} \to \theta_{\rm f}/\theta_{\rm i}$, $y_{\rm f} \to y_{\rm f}/y_{\rm i}$, and $t \to \kappa_{\rm i} t$.

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[26] This is the reason why we employ here the terms deconfining ($\Delta\kappa < 0$) and confining ($\Delta\kappa > 0$) instead of decompressing and compressing. At variance with the isothermal case, the signs of $\Delta\kappa$ and Δy are not directly related in an adiabatic process.

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Supplemental Material for "On the existence and optimisation of finite-time adiabatic processes"

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We present here details pertaining to the framework used, together with a technical and independent rederivation of two key results (optimal temperature for fixed operating time and minimal time for fixed initial and final states), while a calculation-free arguments are provided in the main text.

In section I, we derive the evolution equation for the variance of the Brownian particle position in the overdamped regime, which is our starting point. Moreover, we explicitly show that the corresponding distribution function remains Gaussian for all times. Sections II and III are devoted to the optimisation of finite-time adiabatic processes connecting equilibrium states, which are solved with the help of physical arguments in the letter. Here, we address these problems by employing a rigorous–and thus lengthier–mathematical approach, based on two well-posed variational problems with constraints. By doing so, we obtain in an alternative way the same solutions as in the letter to (i) the optimal temperature for fixed running time, and (ii) the minimum time for fixed initial and final states.

I. EVOLUTION EQUATION FOR THE VARIANCE OF THE POSITION

We consider a Brownian particle immersed in a fluid and trapped in a harmonic potential with stiffness k. The fluid is at equilibrium at temperature T. Note that both the temperature and the stiffness, which are externally controlled, may be time-dependent. In the overdamped limit, the Langevin equation for the particle position $x(t)$ reads

$$\dot{x}(t) = -\frac{k}{\lambda}x(t) + \sqrt{2D}\eta(t), \tag{S1}$$

where λ is the viscous drag, D is the diffusion coefficient, and $\eta(t)$ stands for delta-correlated Gaussian white noise of zero mean and unit variance. The diffusion coefficient verifies the fluctuation-dissipation relation

$$D = \frac{k_B T}{\lambda}, \tag{S2}$$

with k_B the Boltzmann constant. If the temperature is time-dependent, D also depends on time.

The dynamics of the system can also be studied using the probability density function $P(x,t)$ for finding the Brownian particle at position x at time t. Its time evolution is governed by the Fokker-Planck equation

$$\lambda\,\partial_t P(x,t) = k\,\partial_x\left[xP(x,t)\right] + k_B T\,\partial_x^2 P(x,t). \tag{S3}$$

The Langevin equation (S1) and the Fokker-Planck equation (S3) are utterly equivalent and both completely characterise the time evolution of the Brownian particle position–mathematically, the stochastic process [1].

In light of the above, we can obtain the time evolution of all the moments or, alternatively, all the cumulants of the position from either Eq. (S1) or Eq. (S3). If the initial condition $P(x,0)$ is Gaussian, $P(x,t)$ remains Gaussian for all times and the two first cumulants, that is, position's average $\langle x\rangle$ and variance $\sigma^2 = \langle x^2\rangle - \langle x\rangle^2$, completely characterise the evolution of the Brownian particle. This stems from the Gaussian character of the white noise $\eta(t)$ appearing in the Langevin equation, and can also be readily understood from the Fokker-Planck equation by going to Fourier space. This is the route we take in the following.

First, we define the Fourier transform of $P(x,t)$ as

$$G(s,t) \equiv \langle e^{isx} \rangle = \int_{-\infty}^{+\infty} dx\, e^{isx} P(x,t). \tag{S4}$$

Therefore, taking the Fourier transform in Eq. (S3) leads to

$$\lambda\, \partial_t G(s,t) = -k\, s\, \partial_s G(s,t) - k_B T s^2 G(s,t). \tag{S5}$$

On the one hand, the expansion of $G(s,t)$ generates the moments $\mu_n(t) \equiv \langle x^n \rangle(t)$, since $G(s,t) = \sum_{n=0}^{\infty} (is)^n \mu_n(t)/n!$. On the other hand, the expansion of $\ln G(s,t)$ generates the cumulants $\chi_n(t)$,

$$\ln G(s,t) = \sum_{n=1}^{\infty} \frac{(is)^n}{n!} \chi_n(t). \tag{S6}$$

As already said above, the first two cumulants are the mean and the variance of the position, respectively. With the notation we are using, $\chi_1 = \mu_1$ and $\chi_2 = \mu_2 - \mu_1^2$.

Equation (S5) can be rewritten as

$$\lambda\, \partial_t \ln G(s,t) = -k\, s\, \partial_s \ln G(s,t) - k_B T s^2. \tag{S7}$$

Introducing the expansion (S6) into (S7) and equating the coefficients sharing the same power of s, the equations for the cumulants are obtained as

$$\lambda\, \frac{d\chi_n(t)}{dt} = -nk\, \chi_n(t) + 2k_B T\, \delta_{n,2}, \quad n \geq 1. \tag{S8}$$

First, the equation for $n = 1$ implies that the average $\chi_1 = \mu_1 = \langle x \rangle$ remains zero for all times if it is so initially. Second, the equation for $n = 2$ gives the time evolution of the variance $\chi_2 = \mu_2 - \mu_1^2 = \sigma^2$,

$$\lambda\, \frac{d\chi_2}{dt} = -2k\, \chi_2 + 2k_B T. \tag{S9}$$

Third, the equations for $n \geq 2$ entail that an initially Gaussian distribution remains Gaussian for all times: if $\chi_n(0) = 0$ for all $n \geq 2$, we have that $\chi_n(t) = 0$ for all $n \geq 2$. These properties remain valid if k and/or T are time-dependent, since Eqs. (S1)-(S9) hold for both constant and time-dependent k and T. If the stiffness of the trap k and the temperature of the fluid T are time-independent, χ_2 evolves as time increases towards its equilibrium value $\chi_2^{\mathrm{eq}} = k_B T/k$, as predicted by equilibrium statistical mechanics.

Nondimensionalisation is introduced with the definitions

$$\kappa = \frac{k}{k_0}, \quad \theta = \frac{T}{T_0}, \quad y = \frac{\chi_2}{k_B T_0/k_0}, \quad t^* = \frac{t}{\lambda/k_0}, \tag{S10}$$

where k_0 and T_0 are some arbitrary values of the stiffness of the trap and the temperature, respectively. Making use of these dimensionless variables in Eq. (S9), we obtain Eq. (**??**) in the main text–with t^*'s asterisk removed in order not to clutter our notation.

II. OPTIMAL TEMPERATURE FOR FIXED RUNNING TIME

A. Statement of the variational problem

We would like to minimise the final temperature in an adiabatic process for the trapped Brownian particle. Therefore, consider the temperature increment

$$\Delta\theta \equiv \theta_{\mathrm{f}} - \theta_{\mathrm{i}} = \int_0^{t_{\mathrm{f}}} dt\, \dot{\theta}. \tag{S11}$$

This is a "constrained" minimisation problem, since we seek the minimisation of $\Delta\theta$ that is compatible with (i) the time evolution of the variance of the Brownian particle, Eq. (**??**) of the main text,

$$\dot{y} = -2\kappa y + 2\theta, \tag{S12}$$

and (ii) the adiabaticity condition,

$$\kappa\dot{y} + \dot{\theta} = 0, \tag{S13}$$

which is equivalent to Eq. (**??**) of the main text.

Therefore, we have to introduce Lagrange multiplier functions $\lambda(t)$ and $\mu(t)$ ensuring that Eqs. (S12) and (S13) hold for all times, as explained in Ref. [2] for minimisation problems with "auxiliary conditions"–or in Ref. [3] for minimisation problems with "subsidiary conditions". Specifically, we look for functions that make

$$\mathcal{S}[y,\kappa,\theta,\lambda,\mu] = \int_0^{t_\mathrm{f}} dt\,\dot{\theta} + \int_0^{t_\mathrm{f}} dt\,\lambda(t)\left(\dot{y} + 2\kappa y - 2\theta\right) + \int_0^{t_\mathrm{f}} dt\,\mu(t)\left(\kappa\dot{y} + \dot{\theta}\right), \tag{S14}$$

stationary. Then, we have to minimise the "action"

$$\mathcal{S}[y,\kappa,\theta,\lambda,\mu] = \int_0^{t_\mathrm{f}} dt\,\mathcal{L}(\kappa, y, \dot{y}, \theta, \dot{\theta}, \lambda, \mu), \tag{S15}$$

in which we have the "Lagrangian"

$$\mathcal{L}(\kappa, y, \dot{y}, \theta, \dot{\theta}, \lambda, \mu) = \dot{\theta} + \lambda\left(\dot{y} + 2\kappa y - 2\theta\right) + \mu\left(\kappa\dot{y} + \dot{\theta}\right). \tag{S16}$$

Note that the "Lagrangian" does not depend on $\dot{\kappa}$. This means that the corresponding Euler-Lagrange for κ can be used to eliminate κ in favour of the remainder of the variables [4].

The boundary conditions for the minimisation problem are the following:

1. Given initial equilibrium state, that is, given values of κ_i, y_i and θ_i.

$$\boxed{\kappa(t=0) = \kappa_\mathrm{i}, \quad y(t=0) = y_\mathrm{i}, \quad \theta(t=0) = \theta_\mathrm{i} = \kappa_\mathrm{i} y_\mathrm{i}.} \tag{S17}$$

2. Given value of the final stiffness κ_f and equilibrium condition at the final time, $\kappa_\mathrm{f} y_\mathrm{f} = \theta_\mathrm{f}$.

$$\boxed{\kappa(t=t_\mathrm{f}) = \kappa_\mathrm{f}, \quad \kappa(t=t_\mathrm{f}) y(t=t_\mathrm{f}) = \theta(t=t_\mathrm{f}).} \tag{S18}$$

By taking an infinitesimal variation of $\mathcal{S}$ and equating it to zero, not only do we get the Euler-Lagrange equations for the minimisation problem, but also the adequate boundary conditions–as discussed in Ref. [2], section II.15. The boundary term in $\delta\mathcal{S}$ must vanish,

$$p_\kappa \delta\kappa + p_y \delta y + p_\theta \delta\theta\big|_0^{t_\mathrm{f}} = 0. \tag{S19}$$

We have introduced the canonical momenta in the usual way, which for our problem read

$$p_\kappa \equiv \frac{\partial\mathcal{L}}{\partial\dot{\kappa}} = 0, \tag{S20a}$$

$$p_y \equiv \frac{\partial\mathcal{L}}{\partial\dot{y}} = \lambda + \mu\kappa, \tag{S20b}$$

$$p_\theta \equiv \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 1 + \mu. \tag{S20c}$$

Since κ_i, y_i and θ_i are fixed, there is no boundary contribution coming from $t = 0$. For $t = t_\mathrm{f}$, however, we have a different situation, $\delta\kappa_\mathrm{f} = 0$ but y_f and θ_f are simply linked by the equilibrium condition, which entails that $\kappa_\mathrm{f}\delta y_\mathrm{f} = \delta\theta_\mathrm{f}$. Therefore, we have that

$$\cancelto{0}{p_{\kappa\mathrm{f}}\delta\kappa_\mathrm{f}} + p_{y\mathrm{f}}\delta y_\mathrm{f} + p_{\theta\mathrm{f}}\delta\theta_\mathrm{f} = \left(p_{y\mathrm{f}} + p_{\theta\mathrm{f}}\kappa_\mathrm{f}\right)\delta y_\mathrm{f} = 0, \tag{S21}$$

which entails that

$$p_y(t=t_\mathrm{f}) + p_\theta(t=t_\mathrm{f})\kappa(t=t_\mathrm{f}) = 0, \tag{S22}$$

since δy_f is arbitrary. By employing the expressions for p_y and p_θ found above, we get

$$\lambda_\mathrm{f} + \mu_\mathrm{f}\kappa_\mathrm{f} + (1+\mu_\mathrm{f})\kappa_\mathrm{f} = \kappa_\mathrm{f} + \lambda_\mathrm{f} + 2\mu_\mathrm{f}\kappa_\mathrm{f} = 0 \tag{S23}$$

for the lacking boundary condition, that is,

$$\boxed{\kappa(t=t_\mathrm{f}) + \lambda(t=t_\mathrm{f}) + 2\mu(t=t_\mathrm{f})\kappa(t=t_\mathrm{f}) = 0.} \tag{S24}$$

B. Euler-Lagrange equations

Now, we write the Euler-Lagrange equations for the minimisation problem. First, taking into account Eq. (S20a) and $\partial_\kappa \mathcal{L} = 2\lambda y + \mu\dot{y}$,

$$0 = 2\lambda y + \mu\dot{y}. \tag{S25}$$

Second, we bring to bear Eq. (S20b) and $\partial_y \mathcal{L} = 2\kappa\lambda$,

$$\frac{d}{dt}\left(\lambda + \mu\kappa\right) = 2\kappa\lambda. \tag{S26}$$

Third, we make use of Eq. (S20c) and $\partial_\theta \mathcal{L} = -2\lambda$ to write

$$\dot{\mu} = -2\lambda. \tag{S27}$$

In addition, since by construction the Lagrangian does not depend on $\dot{\lambda}$ and $\dot{\mu}$, the Euler-Lagrange equations for λ and μ reduce to the constraints–or auxiliary conditions, given by Eqs. (S12) and (S13).

It is straightforward to get rid of the Lagrange multipliers by first inserting Eq. (S27) into (S25), which gives

$$\mu\dot{y} - y\dot{\mu} = 0 \Rightarrow \quad \boxed{\mu = c_1 y,} \tag{S28}$$

where c_1 is an arbitrary constant, to be determined later by imposing the boundary conditions. Moreover, Eq. (S27) yields

$$\boxed{\lambda = -\frac{c_1}{2}\dot{y}.} \tag{S29}$$

These expressions for the multipliers in terms of y and $\dot{y}$ allow us to work out the solution, as detailed below. The constant c_1 should be non-zero because its vanishing leads to $\lambda(t) = \mu(t) = 0$, that is, the situation without constraints.

Inserting Eqs. (S28) and (S29) into (S26), we get

$$\ddot{y} - 4\kappa\dot{y} - 2\dot{\kappa}y = 0, \tag{S30}$$

after taking into account that $c_1 \neq 0$. By taking time derivative in the evolution equation for y, as given by Eq. (S12), and making use of the adiabatic condition (S13), it is also shown that

$$\ddot{y} + 4\kappa\dot{y} + 2\dot{\kappa}y = 0. \tag{S31}$$

Combining Eqs. (S30) and (S31), we obtain

$$2\kappa\dot{y} + \dot{\kappa}y = 0 \Rightarrow \quad \boxed{\kappa y^2 = c_2,} \tag{S32}$$

where c_2 is an arbitrary constant.

Finally, taking into account Eq. (S32), we find the expressions for the variance and the temperature. The adiabatic condition (S13) is now simplified to

$$c_2\frac{\dot{y}}{y^2} + \dot{\theta} = 0 \Rightarrow \boxed{\theta = \frac{c_2}{y} + \frac{c_3}{2},} \tag{S33}$$

in which c_3 is another arbitrary constant–the factor $1/2$ on the rhs is convenient later. Substituting Eqs. (S32) and (S33) into the evolution equation (S12) gives

$$\dot{y} + 2\frac{\cancel{c_2}}{\cancel{y^2}}y - 2\left(\frac{\cancel{c_2}}{\cancel{y}} + \frac{c_3}{2}\right) = 0 \Rightarrow \dot{y} = c_3, \Rightarrow \boxed{y = c_3 t + c_4.} \tag{S34}$$

Once more, c_4 is an arbitrary constant.

C. Solution of the problem

Equations (S32), (S33) and (S34) provide the solution to the minimisation problem. The constants (c_1, c_2, c_3, c_4) have to be written in terms of physical quantities by imposing the boundary conditions. It may seem odd at first sight that there are 4 constants but 6 boundary conditions. The reason is the same as in other problems in stochastic thermodynamics: κ may have jumps at the boundaries. In the present context, this peculiar behaviour is readily understood: the conjugate moment $p_k = \partial_\kappa \mathcal{L}$ identically vanishes and therefore $\delta\kappa(t=0)$ and $\delta\kappa(t=t_\mathrm{f})$ are in fact arbitrary when imposing the extremality condition $\delta\mathcal{S} = 0$. This means that κ can indeed have finite-jump discontinuities at the initial and final times: $\kappa(t=0^+)$ and $\kappa(t=t_\mathrm{f}^-)$ do not coincide in general with κ_i and κ_f.

Following the discussion above, we now impose the four relevant boundary conditions,

$$y(t=0) = y_\mathrm{i}, \quad \theta(t=0) = \theta_\mathrm{i}, \quad \kappa_\mathrm{f} y(t=t_\mathrm{f}) = \theta(t=t_\mathrm{f}), \quad \kappa_\mathrm{f} + \lambda(t=t_\mathrm{f}) + 2\kappa_\mathrm{f}\mu(t=t_\mathrm{f}) = 0. \tag{S35}$$

The constants c_3 and c_4 are directly obtained as

$$\boxed{c_3 = \frac{y_\mathrm{f} - y_\mathrm{i}}{t_\mathrm{f}}, \quad c_4 = y_\mathrm{i}.} \tag{S36}$$

Note that y_f does not have a definite value but is related to θ_f by the equilibrium condition; this will be brought to bear later. The optimal time evolution for the variance is then

$$\boxed{y(t) = y_\mathrm{i} + \frac{y_\mathrm{f} - y_\mathrm{i}}{t_\mathrm{f}} t.} \tag{S37}$$

Now, particularising Eq. (S33) for $t = 0$ makes it possible to obtain c_2,

$$\theta_\mathrm{i} = \frac{c_2}{y_\mathrm{i}} + \frac{c_3}{2} \Rightarrow \quad \boxed{c_2 = y_\mathrm{i}\left(\theta_\mathrm{i} - \frac{y_\mathrm{f} - y_\mathrm{i}}{2t_\mathrm{f}}\right).} \tag{S38}$$

Using again Eq. (S33) but for an arbitrary time t, after some simple algebra one gets

$$\boxed{\theta(t) = \frac{y_\mathrm{i}\theta_\mathrm{i} + \dfrac{(y_\mathrm{f} - y_\mathrm{i})^2}{2t_\mathrm{f}^2} t}{y_\mathrm{i} + \dfrac{y_\mathrm{f} - y_\mathrm{i}}{t_\mathrm{f}} t}.} \tag{S39}$$

Substituting $t = t_f$ into this equation, we obtain

$$\theta_\mathrm{f} = \frac{y_\mathrm{i}\theta_\mathrm{i}}{y_\mathrm{f}} + \frac{(y_\mathrm{f} - y_\mathrm{i})^2}{2t_\mathrm{f} y_\mathrm{f}}, \tag{S40}$$

so that

$$\theta_\mathrm{f} \geq \frac{y_\mathrm{i}\theta_\mathrm{i}}{y_\mathrm{f}}, \tag{S41}$$

with the equality holding in the limit as $t_\mathrm{f} \to \infty$, that is, in the quasi-static limit.

We have yet to impose the boundary condition $y_\mathrm{f} = \theta_\mathrm{f}/\kappa_\mathrm{f}$. We do so in Eq. (S40),

$$\theta_\mathrm{f} = \frac{\kappa_\mathrm{f} y_\mathrm{i}\theta_\mathrm{i}}{\theta_\mathrm{f}} + \frac{\kappa_\mathrm{f}}{2t_\mathrm{f}} \frac{\left(\dfrac{\theta_\mathrm{f}}{\kappa_\mathrm{f}} - y_\mathrm{i}\right)^2}{\theta_\mathrm{f}} \Rightarrow \theta_\mathrm{f}^2 = \kappa_\mathrm{f} y_\mathrm{i}\theta_\mathrm{i} + \frac{\kappa_\mathrm{f}}{2t_\mathrm{f}}\left(\frac{\theta_\mathrm{f}}{\kappa_\mathrm{f}} - y_\mathrm{i}\right)^2. \tag{S42}$$

This equation is a quadratic equation for θ_f in terms of the fixed parameters κ_f, y_i, θ_i, and t_f. Also, it can be rewritten as

$$\boxed{\left(1 - \frac{1}{2\kappa_\mathrm{f} t_\mathrm{f}}\right)\theta_\mathrm{f}^2 + \frac{y_\mathrm{i}}{t_\mathrm{f}}\theta_\mathrm{f} - \kappa_\mathrm{f} y_\mathrm{i}\left(\theta_\mathrm{i} + \frac{y_i}{2t_\mathrm{f}}\right) = 0,} \tag{S43}$$

which is equivalent to Eq. (**??**) for $\tilde{\theta}_{\rm f}$ in the main text, after taking into account that $y_{\rm i} = \theta_{\rm i}/\kappa_{\rm i}$.

It is worth noting that the constant c_1 has not been necessary to obtain the solution for the physical quantities, the stiffness $\kappa(t)$, the variance $y(t)$, and the temperature $\theta(t)$. It is only needed to derive the final expressions for the Lagrange multipliers $\lambda(t)$ and $\mu(t)$. For the sake of completeness, we give the expression for c_1 that follows from Eq. (S24),

$$\kappa_{\rm f} - \frac{c_1 c_3}{2} + 2c_1 y_{\rm f}\kappa_{\rm f} = 0 \Rightarrow \quad c_1 = \frac{2\kappa_{\rm f}}{c_3 - 4\theta_{\rm f}}. \tag{S44}$$

III. MINIMUM TIME FOR FIXED INITIAL AND FINAL STATES

We turn our attention to another optimisation problem: obtain the minimum time to connect two given equilibrium states with an adiabatic process. This problem has been solved in the main text by an ad-hoc procedure, but it can be addressed in a way similar to the one employed in the previous section. In this case, we would like to minimise

$$t_{\rm f} = \int_0^{t_{\rm f}} dt\, 1, \tag{S45}$$

submitted again to the constraints given by Eq. (S12) and Eq. (S13). Therefore, we have to minimise a new "action"

$$\hat{\mathcal{S}}[y,\kappa,\theta,\lambda,\mu] = \int_0^{t_{\rm f}} dt\, \hat{\mathcal{L}}(\kappa, y, \dot{y}, \theta, \dot{\theta}, \lambda, \mu), \tag{S46}$$

in which we have the new "Lagrangian"

$$\begin{aligned}\hat{\mathcal{L}}(\kappa, y, \dot{y}, \theta, \dot{\theta}, \lambda, \mu) &= 1 + \lambda\left(\dot{y} + 2\kappa y - 2\theta\right) + \mu\left(\kappa\dot{y} + \dot{\theta}\right) = \mathcal{L}(\kappa, y, \dot{y}, \theta, \dot{\theta}, \lambda, \mu) + 1 - \dot{\theta} \\ &= \mathcal{L}(\kappa, y, \dot{y}, \theta, \dot{\theta}, \lambda, \mu) + \frac{d}{dt}\left(t - \theta\right).\end{aligned} \tag{S47}$$

Since $\hat{\mathcal{L}}$ and $\mathcal{L}$ differ by the total derivative of a function that depend only on the "coordinates"–and not on the velocities, we know that the Euler-Lagrange equations for both minimisation problems will be the same. Anyhow, we cannot yet conclude that the solution to both problems is the same, since the boundary conditions for them are not [5].

In this case, the boundary conditions are simpler than those addressed in section II, because (κ, y, θ) have prescribed values at the initial and final times, although the latter is not fixed; it is the quantity that we want to minimise. Specifically, Eq. (S17) and Eq. (S18) remain valid but Eq. (S24) must be substituted with

$$\theta(t = t_{\rm f}) = \theta_{\rm f}. \tag{S48}$$

Therefore, we deal with a "standard" variational problem, for which $\delta\kappa$, δy and $\delta\theta$ vanish at the boundaries, similarly to the situation found in Classical Mechanics. Notwithstanding, once more we have that κ may have finite jump discontinuities at the boundaries, recall that its corresponding canonical momentum verifies $\hat{p}_\kappa \equiv 0$.

Since the Euler-Lagrange equations are unchanged, Eq. (S28), Eq. (S29), Eq. (S32), Eq. (S33) and Eq. (S34) still hold. In principle, we should have to reobtain the constants (c_2, c_3, c_4) with the new boundary conditions. However, it is readily realised that the substitution of Eq. (S24) with Eq. (S48) leaves their expressions unchanged, because Eq. (S24) was not employed in their derivation for the optimal temperature problem. The only difference is that $\theta_{\rm f}$ is now fixed and $t_{\rm f}$ is the variable being minimised, instead of the other way round.

In light of the discussion above, it appears that the same function relates the optimal values $\theta_{\rm f}$ and $t_{\rm f}$ for both physical situations, as argued in the main text on physical grounds. Therefore, solving Eq. (S42) for $t_{\rm f}$ and writing it in terms of the initial and final values of θ and κ, we obtain

$$t_{\rm f} = \frac{\left(\dfrac{\theta_{\rm f}}{\kappa_{\rm f}} - \dfrac{\theta_{\rm i}}{\kappa_{\rm i}}\right)^2}{2\left(\dfrac{\theta_{\rm f}^2}{\kappa_{\rm f}} - \dfrac{\theta_{\rm i}^2}{\kappa_{\rm i}}\right)}, \tag{S49}$$

which is the expression for $\tilde{t}_{\rm f}$ in Eq. (**??**) of the main text.

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[1] N. G. Van Kampen, *Stochastic processes in Physics and Chemistry* (North-Holland, 1992).

[2] C. Lanczos, *The variational principles of mechanics* (Dover Publications, 1970).
[3] I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Dover Publications, 2000).
[4] The same is true for the Lagrange multipliers λ and μ, but this is not a peculiarity of the problem with which we deal here, but a general property of the Lagrange multiplier method: by construction, the Lagrangian never depends on the time derivatives of the Lagrangian multipliers.
[5] In this particular case, the fact that the Euler-Lagrange equations do not change can be easily seen, without "invoking" the equivalence of Lagrangians differing by the time derivative of a function. The equality $\hat{\mathcal{L}} = \mathcal{L} + 1 - \dot{\theta}$ implies that all the partial derivatives of $\hat{\mathcal{L}}$ and $\mathcal{L}$ are equal, except for the momentum $\hat{p}_\theta \equiv \partial\hat{\mathcal{L}}/\partial\dot{\theta} = p_\theta - 1$. However, since it is the derivative of the momenta that enter into the Euler-Lagrange equations, they obviously remain unchanged.