# Shortcuts to adiabaticity

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## Abstract

Quantum adiabatic processes –that keep constant the populations in the instantaneous eigenbasis of a time-dependent Hamiltonian– are very useful to prepare and manipulate states, but take typically a long time. This is often problematic because decoherence and noise may spoil the desired final state, or because some applications require many repetitions. "Shortcuts to adiabaticity" are alternative fast processes which reproduce the same final populations, or even the same final state, as the adiabatic process in a finite, shorter time. Since adiabatic processes are ubiquitous, the shortcuts span a broad range of applications in atomic, molecular and optical physics, such as fast transport of ions or neutral atoms, internal population control and state preparation (for nuclear magnetic resonance or quantum information), cold atom expansions and other manipulations, cooling cycles, wavepacket splitting, and many-body state engineering or correlations microscopy. Shortcuts are also relevant to clarify fundamental questions such as a precise quantification of the third principle of thermodynamics and quantum speed limits. We review different theoretical techniques proposed to engineer the shortcuts, the experimental results, and the prospects.

*Key words:* adiabatic dynamics, quantum speed limits, superadiabaticity,

quantum state engineering, transport engineering of cold atoms, ions, and
Bose-Einstein condensates, wave packet splitting, third principle of
thermodynamics, transitionless tracking algorithm, fast expansions

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**Contents**

# 1   Introduction

The expression "shortcuts to adiabaticity" (STA) was recently introduced in Chen et al. (2010b), to describe protocols that speed up a quantum adiabatic process, usually, although not necessarily, through a non-adiabatic route. [1] There the Lewis-Riesenfeld invariants were used to inverse engineer the time dependence of a harmonic oscillator frequency between predetermined initial and final values so as to avoid final excitations. That paper and its companion on Bose-Einstein condensates (Muga et al., 2009) have indeed triggered a surge of activity, not only for harmonic expansions (Chen and Muga, 2010; Muga et al., 2010; Stefanatos et al., 2010; Schaff et al., 2010, 2011a; del Campo, 2011a; Schaff et al., 2011b; Stefanatos et al., 2011; Torrontegui et al., 2012c; Fasihi et al., 2012; Torrontegui et al., 2012a; del Campo and Boshier, 2012; Stefanatos and Li, 2012), but for atom transport (Torrontegui et al., 2011, 2012d; Chen et al., 2011b; Bowler et al., 2012), quantum computing (Sarandy et al., 2011), quantum simulations (Lau and James, 2012), optical lattice expansions (Yuce, 2012; Ozcakmakli and Yuce, 2012), wavepacket splitting (Torrontegui et al., 2012b), internal state control (Chen et al., 2011a; Ibáñez et al., 2011; Ruschhaupt et al., 2012; Ban et al., 2012; Ibáñez et al., 2012a; Güngördü et al., 2012), many-body state engineering (del Campo, 2011b; del Campo and Boshier, 2012; del Campo et al., 2012; Juliá-Díaz et al., 2012), and other applications such as sympathetic cooling of atomic mixtures (Choi, Onofrio and Sundaram, 2011; Choi et al., 2012), or cooling of nanomechanical resonators (Li et al., 2011; Zhang et al., 2012a). In fact several works had previously or simultaneously considered to speed up adiabatic processes making use of different techniques. For example Demirplak and Rice (2003, 2005, 2008) and Berry (2009) proposed the addition of counterdiabatic terms to a reference Hamiltonian $H_0$ to achieve adiabatic dynamics with respect to $H_0$. This "transitionless tracking algorithm" (Berry, 2009) has been applied to manipulate

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[1]   The word "adiabatic" may have two different meanings: the thermodynamical one (no heat transfer between system and environment) and the quantum one, as stated by Born and Fock (1928) in the adiabatic theorem: "a physical system remains in its instantaneous eigenstate when a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum". Here we shall always understand "adiabatic" in the quantum mechanical sense.

the populations of two-level systems (Demirplak and Rice, 2005; Berry, 2009; Chen et al., 2010c; Bason et al., 2012; Zhang et al., 2012b). Another technique to design laser pulses for fast population transfer is parallel adiabatic passage (Gulérin et al., 2002; Vasilev et al., 2009; Dridi et al., 2009; Gulérin et al., 2011). Couvert et al. (2008a) designed trap motions in order to perform non-adiabatic fast transport of atomic cold clouds. Also, Masuda and Nakamura (2010) developed a "fast-forward technique" for several manipulations on wavepackets such as expansions, transport or splitting of Bose-Einstein condensates. Related work had also been carried out for wave packet splitting making use of optimal control (Hohenester et al., 2007; Grond et al., 2009a,b), and in the context of quantum refrigerators, to find fast "frictionless" expansions (Salamon et al., 2009; Rezek et al., 2009). For recent developments on this line stimulated by invariant-based engineering results see Kosloff and Feldmann (2010); Hoffmann et al. (2011); Levy and Kosloff (2012); Feldmann and Kosloff (2012).

The considerable number of publications on the subject, and a recent Conference on "Shortcuts to adiabaticity" held in Bilbao (16-20 July 2012) demonstrate much current interest, not only within the cold atoms and atomic physics communities but also from fields such as semiconductor physics and spintronics (Ban et al., 2012). Indeed adiabatic processes are ubiquitous, so we may expect a broad range of applications, even beyond the quantum domain, since some of the concepts are easy to translate into optics (Lin et al., 2012; Tseng and Chen, 2012) or mechanics (Ibáñez et al., 2011). Apart from the practical applications, the fundamental implications of shortcuts on quantum speed limits (Bender et al., 2007; Bason et al., 2012; Uzdin et al., 2012), time-energy uncertainty relations (Chen and Muga, 2010), multiple Schrödinger pictures (Ibáñez et al., 2012a), and the quantification of the third principle of thermodynamics and of maximal cooling rates (Salamon et al., 2009; Rezek et al., 2009; Chen and Muga, 2010; Kosloff and Feldmann, 2010; Hoffmann et al., 2011; Levy and Kosloff, 2012; Feldmann and Kosloff, 2012) are also intriguing and provide further motivation.

In this review we shall first describe different approaches to STA in Sec. 2. While the main goal there is to construct new protocols for a fast manipulation of quantum states avoiding final excitations, additional conditions may be imposed. For example, ideally these protocols should not be state specific but work for an arbitrary state[2]. They should also be stable against perturbations, and keep the values of the transient energy and other variables manageable throughout the whole process. Several applications are discussed

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[2]  Contrast this to the quantum brachistochrone (Bender et al., 2007), in which the aim is to find a time-independent Hamiltonian that takes a given initial state to a given final state in minimal time. Studies of "quantum speed limits" adopt in general this state-to-state approach, as in Uzdin et al. (2012).

in Secs. 3 to 6. We have kept a notation consistency within each Section but not throughout the whole review, following when possible notations close to the original publications.

## 2  General Formalisms

### 2.1  Invariant Based Inverse Engineering

*Lewis-Riesenfeld invariants.* – The Lewis and Riesenfeld (1969) theory is applicable to a quantum system that evolves with a time-dependent Hermitian Hamiltonian $H(t)$, which supports a Hermitian dynamical invariant $I(t)$ satisfying

$$i\hbar \frac{\partial I(t)}{\partial t} - [H(t), I(t)] = 0. \tag{1}$$

Therefore its expectation values for an arbitrary solution of the time-dependent Schrödinger equation $i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle$, do not depend on time. $I(t)$ can be used to expand $|\Psi(t)\rangle$ as a superposition of "dynamical modes" $|\psi_n(t)\rangle$,

$$|\Psi(t)\rangle = \sum_n c_n |\psi_n(t)\rangle, \quad |\psi_n(t)\rangle = e^{i\alpha_n(t)}|\phi_n(t)\rangle, \tag{2}$$

where $n = 0, 1, ...$; $c_n$ are time-independent amplitudes, and $|\phi_n(t)\rangle$ are orthonormal eigenvectors of the invariant $I(t)$,

$$I(t) = \sum_n |\phi_n(t)\rangle \lambda_n \langle \phi_n(t)|. \tag{3}$$

The $\lambda_n$ are real constants, and the Lewis-Riesenfeld phases are defined as (Lewis and Riesenfeld, 1969)

$$\alpha_n(t) = \frac{1}{\hbar} \int_0^t \left\langle \phi_n(t') \middle| i\hbar \frac{\partial}{\partial t'} - H(t') \middle| \phi_n(t') \right\rangle dt'. \tag{4}$$

We use for simplicity a notation for a discrete spectrum of $I(t)$ but the generalization to a continuum or mixed spectrum is straightforward. We also assume a non-degenerate spectrum. Non-Hermitian invariants and Hamiltonians have been considered for example in Gao et al. (1991, 1992); Lohe (009); Ibáñez et al. (2011).

*Inverse engineering.* – Suppose that we want to drive the system from an initial Hamiltonian $H(0)$ to a final one $H(t_f)$, in such a way that the populations in the initial and final instantaneous bases are the same, but admitting transitions at intermediate times. To inverse engineer a time-dependent Hamiltonian $H(t)$ and achieve this goal, we may first define the invariant through its eigenvalues and eigenvectors. The Lewis-Riesenfeld phases $\alpha_n(t)$ may also

be chosen as arbitrary functions to write down the time-dependent unitary evolution operator $U$,

$$U = \sum_n e^{i\alpha_n(t)}|\phi_n(t)\rangle\langle\phi_n(0)|. \tag{5}$$

$U$ obeys $i\hbar\dot{U} = H(t)U$, where the dot means time derivative. Solving formally this equation for $H(t) = i\hbar\dot{U}U^\dagger$, we get

$$H(t) = -\hbar\sum_n |\phi_n(t)\rangle\dot{\alpha}_n\langle\phi_n(t)| + i\hbar\sum_n |\partial_t\phi_n(t)\rangle\langle\phi_n(t)|. \tag{6}$$

According to Eq. (6), for a given invariant there are many possible Hamiltonians corresponding to different choices of phase functions $\alpha_n(t)$. In general $I(0)$ does not commute with $H(0)$, so the eigenstates of $I(0)$, $|\phi_n(0)\rangle$, do not coincide with the eigenstates of $H(0)$. $H(t_f)$ does not necessarily commute with $I(t_f)$ either. If we impose $[I(0), H(0)] = 0$ and $[I(t_f), H(t_f)] = 0$, the eigenstates will coincide, which guarantees a state transfer without final excitations. In typical applications the Hamiltonians $H(0)$ and $H(t_f)$ are given, and set the initial and final configurations of the external parameters. Then we define $I(t)$ and its eigenvectors accordingly, so that the commutation relations are obeyed at the boundary times and, finally, $H(t)$ is designed via Eq. (6). While the $\alpha_n(t)$ may be taken as fully free time-dependent phases in principle, they may also be constrained by a pre-imposed or assumed structure of $H(t)$. Secs. 3, 4 and 5 present examples of how this works for expansions, transport and internal state control.

A generalization of this inverse method for non-Hermitian Hamiltonians was considered in Ibáñez et al. (2011). Inverse engineering was applied to accelerate the slow expansion of a classical particle in a time-dependent harmonic oscillator without final excitation. This system may be treated formally as a quantum two-level system with non-Hermitian Hamiltonian (Gao et al., 1991, 1992).

*Quadratic in momentum invariants.* – Lewis and Riesenfeld (1969) paid special attention to the time-dependent harmonic oscillator and its invariants quadratic in position and momentum. Later on Lewis and Leach (1982) found, in the framework of classical mechanics, the general form of the Hamiltonian compatible with quadratic-in-momentum invariants, which includes non harmonic potentials. This work, and the corresponding quantum results of Dhara and Lawande (1984) constitutes the basis of this subsection.

A one-dimensional Hamiltonian with a quadratic-in-momentum invariant must have the form $H = p^2/2m + V(q,t)$,[3] with the potential (Lewis and Leach,

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[3]  $q$ and $p$ may denote operators or numbers. The context should clarify their exact meaning.

1982; Dhara and Lawande, 1984)

$$V(q,t) = -F(t)q + \frac{m}{2}\omega^2(t)q^2 + \frac{1}{\rho(t)^2}U\left[\frac{q - q_c(t)}{\rho(t)}\right]. \tag{7}$$

$\rho$, $q_c$, $\omega$, and $F$ are arbitrary functions of time that satisfy the auxiliary equations

$$\ddot{\rho} + \omega^2(t)\rho = \frac{\omega_0^2}{\rho^3}, \tag{8}$$

$$\ddot{q}_c + \omega^2(t)q_c = F(t)/m, \tag{9}$$

where $\omega_0$ is a constant. Their physical interpretation will be explained below and depends on the operation. A quadratic-in-$p$ dynamical invariant is given, up to a constant factor, by

$$I = \frac{1}{2m}[\rho(p - m\dot{q}_c) - m\dot{\rho}(q - q_c)]^2 + \frac{1}{2}m\omega_0^2\left(\frac{q - q_c}{\rho}\right)^2 + U\left(\frac{q - q_c}{\rho}\right). \tag{10}$$

Now $\alpha_n$ in Eq. (4) satisfies (Lewis and Riesenfeld, 1969; Dhara and Lawande, 1984)

$$\alpha_n = -\frac{1}{\hbar}\int_0^t dt'\left(\frac{\lambda_n}{\rho^2} + \frac{m(\dot{q}_c\rho - q_c\dot{\rho})^2}{2\rho^2}\right), \tag{11}$$

and the function $\phi_n$ can be written as (Dhara and Lawande, 1984)

$$\phi_n(q,t) = e^{\frac{im}{\hbar}[\dot{\rho}q^2/2\rho + (\dot{q}_c\rho - q_c\dot{\rho})q/\rho]}\frac{1}{\rho^{1/2}}\Phi_n\left(\underbrace{\frac{q - q_c}{\rho}}_{=:\sigma}\right) \tag{12}$$

in terms of the solution $\Phi_n(\sigma)$ (normalized in $\sigma$-space) of the auxiliary Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\sigma^2} + \frac{1}{2}m\omega_0^2\sigma^2 + U(\sigma)\right]\Phi_n = \lambda_n\Phi_n. \tag{13}$$

The strategy of invariant-based inverse engineering here is to design $\rho$ and $q_c$ first so that $I$ and $H$ commute at initial and final times, except for launching or stopping atoms as in Torrontegui et al. (2011). Then $H$ is deduced from Eq. (7). Applications will be discussed in Secs. 3 and 4.

## 2.2 Counterdiabatic or Transitionless Tracking Approach

For the transitionless driving or counterdiabatic approach as formulated by Berry (2009), and equivalently by Demirplak and Rice (2003, 2005, 2008)[4], the starting point is a time-dependent reference Hamiltonian,

$$H_0(t) = \sum_n |n_0(t)\rangle E_n^{(0)}(t)\langle n_0(t)|. \tag{14}$$

The approximate time-dependent adiabatic solution of the dynamics with $H_0$ takes the form

$$|\psi_n^{(ad)}(t)\rangle = e^{i\xi_n(t)}|n_0(t)\rangle, \tag{15}$$

where the adiabatic phase reads

$$\xi_n(t) = -\frac{1}{\hbar}\int_0^t dt' E_n^{(0)}(t') + i\int_0^t dt' \langle n_0(t')|\partial_{t'} n_0(t')\rangle. \tag{16}$$

The approximate adiabatic vectors in Eq. (15) are defined differently from the dynamical modes of the previous subsection, but they may potentially coincide, as we shall see. Defining now the unitary operator

$$U = \sum_n e^{i\xi_n(t)}|n_0(t)\rangle\langle n_0(0)|, \tag{17}$$

a Hamiltonian $H(t) = i\hbar \dot{U}U^\dagger$ can be constructed to drive the system exactly along the adiabatic paths of $H_0(t)$, as $H(t) = H_0(t) + H_{cd}(t)$, where

$$H_{cd}(t) = i\hbar \sum_n \left( |\partial_t n_0(t)\rangle\langle n_0(t)| - \langle n_0(t)|\partial_t n_0(t)\rangle|n_0(t)\rangle\langle n_0(t)| \right) \tag{18}$$

is purely non-diagonal in the $\{|n_0(t)\rangle\}$ basis.

We may change the $E_n^{(0)}(t)$, and therefore $H_0(t)$ itself, keeping the same $|n_0(t)\rangle$. We could for example make all the $E_n^{(0)}(t)$ zero, or set $\xi_n(t) = 0$ (Berry, 2009). Taking into account this freedom the Hamiltonian for transitionless driving can be generally written as

$$H(t) = -\hbar \sum_n |n_0(t)\rangle\dot{\xi}_n\langle n_0(t)| + i\hbar \sum_n |\partial_t n_0(t)\rangle\langle n_0(t)|. \tag{19}$$

Subtracting $H_{cd}(t)$, the generic $H_0$ is

$$H_0(t) = \sum_n |n_0(t)\rangle\left[i\hbar\langle n_0(t)|\partial_t n_0(t)\rangle - \hbar\dot{\xi}_n\right]\langle n_0(t)|. \tag{20}$$

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[4] Berry's transitionless driving method is equivalent to the counterdiabatic approach of Demirplak and Rice (2003, 2005, 2008). In Section 2.4 we shall see how to further exploit this scheme together with "superadiabatic iterations".

It is usually required that $H_{cd}(t)$ vanish for $t < 0$ and $t > t_f$, either suddenly or continuously at the boundary times. In that case the $\{|n_0(t)\rangle\}$ become also at the extreme times (at least at $t = 0^-$ and $t = t_f^+$) eigenstates of the full Hamiltonian.

Using Eq. (1) and the orthonormality of the $\{|n_0(0)\rangle\}$ we may write invariants of $H(t)$ with the form $I(t) = \sum_n |n_0(t)\rangle \lambda_n \langle n_0(t)|$. For the simple choice $\lambda_n = E_n^{(0)}(0)$, then $I(0) = H_0(0)$.

In this part and also in Sec. 2.1 the invariant-based and transitionless-tracking-algorithm approaches have been presented in a common language to make their relations obvious. Reinterpreting the phases of Berry's method as $\xi_n(t) = \alpha_n(t)$, and the states as $|n_0(t)\rangle = |\phi_n(t)\rangle$, the Hamiltonians $H(t)$ in Eqs. (6) and (19) may be equated. As well, the $H_0(t)$ implicit in the invariant-based method is given by Eq. (20), so that the dynamical modes can be also understood as approximate adiabatic modes of $H_0(t)$ (Chen et al., 2011a). An important caveat is that the two methods could coincide but they do not have to. Given $H(0)$ and $H(t_f)$, there is much freedom to interpolate them using different invariants, phase functions, and reference Hamiltonians $H_0(t)$. In other words, these methods do not provide a unique shortcut but families of them. This flexibility enables us to optimize the path according to physical criteria and/or operational constraints.

*Non-Hermitian Hamiltonians.–* A generalization is possible for non-Hermitian Hamiltonians in a weak non-hermiticity regime (Ibáñez et al., 2011, 2012b). It was applied to engineer a shortcut laser interaction and accelerate the decay of a two-level atom with spontaneous decay. Note that the concept of "population" is problematic for non-Hermitian Hamiltonians (Leclerc et al., 2012). This affects in particular the definition of "adiabaticity" and of the shortcut concept. It is useful to rely instead on normalization-independent quantities, such as the norm of a wave-function component in a biorthogonal basis (Ibáñez et al., 2012b).

*Many-body Systems.–* Following del Campo et al. (2012), the transitionless quantum driving can be extended as well to many-body quantum critical systems, exploiting recent advances in the simulation of coherent $k$-body interactions (Müller et al., 2011; Barreiro et al., 2011). In this context STA allow a finite-rate crossing of a second order quantum phase transition without creating excitations. Consider the family of quasi-free fermion Hamiltonians in dimension $D$, $\mathcal{H}_0 = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger [\vec{a}_{\mathbf{k}}(\lambda(t)) \cdot \vec{\sigma}_{\mathbf{k}}] \psi_{\mathbf{k}}$, where the $\mathbf{k}$-mode Pauli matrices are $\vec{\sigma}_{\mathbf{k}} \equiv (\sigma_{\mathbf{k}}^x, \sigma_{\mathbf{k}}^y, \sigma_{\mathbf{k}}^z)$ and $\psi_{\mathbf{k}}^\dagger = (c_{\mathbf{k},1}^\dagger, c_{\mathbf{k},2}^\dagger)$ are fermionic operators, and the sum goes over independent $\mathbf{k}$-modes. Particular instances of quantum critical models within this family of Hamiltonians are the Ising and XY models in $D = 1$ (Sachdev, 1999), and the Kitaev model in $D = 2$ (Lee et al., 2007) and $D = 1$ (Sengupta et al., 2008). The function $\vec{a}_{\mathbf{k}}(\lambda) \equiv (a_{\mathbf{k}}^x(\lambda), a_{\mathbf{k}}^y(\lambda), a_{\mathbf{k}}^z(\lambda))$

is specific for each model (Dziarmaga et al., 2010). All these models can be written down as a sum of independent Landau-Zener crossings, where the instantaneous $\mathbf{k}$-mode eigenstates have eigenenergies $\varepsilon_{\mathbf{k},\pm} = \pm|\vec{a}_{\mathbf{k}}(\lambda)| = \pm\sqrt{a_{\mathbf{k}}^x(\lambda)^2 + a_{\mathbf{k}}^y(\lambda)^2 + a_{\mathbf{k}}^z(\lambda)^2}$. It is possible to adiabatically cross the quantum critical point driving the dynamics along the instantaneous eigenmodes of $\mathcal{H}_0$ provided that the dynamics is driven by the modified Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{cd}$, where (del Campo et al., 2012)

$$\mathcal{H}_{cd} = \lambda'(t) \sum_{\mathbf{k}} \frac{1}{2|\vec{a}_{\mathbf{k}}(\lambda)|^2} \psi_{\mathbf{k}}^\dagger \left[ (\vec{a}_{\mathbf{k}}(\lambda) \times \partial_\lambda \vec{a}_{\mathbf{k}}(\lambda)) \cdot \vec{\sigma}_{\mathbf{k}} \right] \psi_{\mathbf{k}} \qquad (21)$$

is typically highly non-local in real spaces and involves many-body interactions in the spin representation. However, it was shown in the 1D quantum Ising model that a truncation of $\mathcal{H}_{cd}$ with interactions restricted to range $M$ is efficient to suppress excitations on modes $k > M^{-1}$ (del Campo et al., 2012).

## 2.3   Fast-forward Approach

Based on some earlier results (Masuda and Nakamura, 2008), the fast-forward (FF) formalism for adiabatic dynamics and application examples were worked out in Masuda and Nakamura (2010, 2011); Masuda (2012) for the Gross-Pitaevskii equation or the corresponding Schrödinger equation. The aim of the method is to accelerate a "standard" system subjected to a slow variation of external parameters by canceling a divergence due to an infinitely-large magnification factor with the infinitesimal slowness due to adiabaticity. A fast-forward potential is constructed which leads to the speeded-up evolution but, as a consequence of the different steps and functions introduced, the method is somewhat involved, which possibly hinders a broader application. The streamlined construction of fast-forward potentials presented in Torrontegui et al. (2012a) is followed here.

The starting point is the 3D time-dependent Gross-Pitaevskii (GP) equation (Dalfovo et al., 1999)

$$i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t) + V(\mathbf{x}, t)\psi(\mathbf{x}, t) + g_3|\psi(\mathbf{x}, t)|^2 \psi(\mathbf{x}, t). \qquad (22)$$

Using the ansatz $\psi(\mathbf{x}, t) = r(\mathbf{x}, t)e^{i\phi(\mathbf{x}, t)}$ ($r(\mathbf{x}, t), \phi(\mathbf{x}, t) \in \mathbb{R}$) we formally solve for $V(\mathbf{x}, t)$ in (22) and get for the real and imaginary parts

$$\text{Re}[V(\mathbf{x}, t)] = -\hbar\dot{\phi} + \frac{\hbar^2}{2m}\left(\frac{\nabla^2 r}{r} - (\nabla\phi)^2\right) - g_3 r^2, \tag{23}$$

$$\text{Im}[V(\mathbf{x}, t)] = \hbar\frac{\dot{r}}{r} + \frac{\hbar^2}{2m}\left(\frac{2\nabla\phi \cdot \nabla r}{r} + \nabla^2\phi\right). \tag{24}$$

Imposing $\text{Im}[V(\mathbf{x}, t)] = 0$, i.e.,

$$\frac{\dot{r}}{r} + \frac{\hbar}{2m}\left(\frac{2\nabla\phi \cdot \nabla r}{r} + \nabla^2\phi\right) = 0, \tag{25}$$

Eq. (23) gives a real potential. In the inversion protocol it is assumed that the full Hamiltonian and the corresponding eigenstates are known at the boundary times. Then we design $r(\mathbf{x}, t)$, solve for $\phi$ in Eq. (25), and finally get the potential $V$ from Eq. (23). In Torrontegui et al. (2012a) it was shown how the work of Masuda and Nakamura (2008, 2010, 2011) relates to this streamlined construction.

Since the phase $\phi$ that solves Eq. (25) depends in general on the particular $r(\mathbf{x}, t)$, Eq. (23) gives in principle a state-dependent potential. However, in some special circumstances, the fast-forward potential remains the same for all modes. This happens in particular for the Schrödinger equation, $g_3 = 0$, and Lewis-Leach potentials associated with quadratic-in-momentum invariants. In other words, the invariant-based approach can be formulated as a special case of the simple inverse method (Torrontegui et al., 2012a).

## 2.4 Alternative Shortcuts Through Unitary Transformations

Shortcuts found via the methods described so far or by any other approach might be difficult to implement in practice. In the cd approach, for instance, the structure of the complementary Hamiltonian $H_{cd}$ could be quite different from the structure of the reference Hamiltonian $H_0$. Here are three examples, the first two for a particle of mass $m$ in 1D, the third one for a two-level system:

- Example 1: Harmonic oscillator expansions (Muga et al., 2010), see Sec. 3:

$$H_0 = p^2/(2m) + m\omega^2 q^2/2, \quad H_{cd} = -(pq + qp)\dot{\omega}/(4\omega). \tag{26}$$

- Example 2: Harmonic transport with a trap of constant frequency $\omega_0/2\pi$ and displacement $q_0(t)$ (Torrontegui et al., 2011), see Sec. 4:

$$H_0 = p^2/(2m) + (q - q_0(t))^2 m\omega_0^2/2, \quad H_{cd} = p\dot{q}_0. \tag{27}$$

- Example 3: Population inversion in a two-level system (Berry, 2009; Chen et al., 2010c; Ibáñez et al., 2012a), see Sec. 5:

$$H_0 = \begin{pmatrix} Z_0 & X_0 \\ X_0 & -Z_0 \end{pmatrix}, \quad H_{cd} = \hbar(\dot{\Theta}_0/2)\sigma_y, \tag{28}$$

where $\Theta_0 = \arccos(Z_0/R_0)$ is a polar angle and $R_0 = (X_0^2 + Z_0^2)^{1/2}$.

In all these examples the experimental implementation of $H_0$ is possible, but the realization of the counter-diabatic terms is problematic. A way out is provided by unitary transformations that generate alternative shortcut protocols without the undesired terms in the Hamiltonian (Ibáñez et al., 2012a). A standard tool is the use of different interaction pictures for describing one physical setting. Unitary operators $\mathcal{U}(t)$ connect the different pictures and the goal is frequently to work in a picture that facilitates the mathematical manipulations. In this standard scenario all pictures describe the same physics, the same physical experiments and manipulations.

The main idea in Ibáñez et al. (2012a) is to regard instead the unitary transformations as a way to generate different physical settings and different experiments, not just as mathematical transformations. The starting point is a shortcut described by the Schrödinger equation $i\hbar\partial_t\psi(t) = H(t)\psi(t)$, our reference protocol. (In all the above examples $H = H_0 + H_{cd}$.) The new dynamics is given by $i\hbar\partial_t\psi'(t) = H'(t)\psi'(t)$, where $\psi'(t) = \mathcal{U}(t)^\dagger\psi(t)$, and $H' = \mathcal{U}^\dagger(H - K)\mathcal{U}$, where $K = i\hbar\dot{\mathcal{U}}\mathcal{U}^\dagger$. If $\mathcal{U}(0) = \mathcal{U}(t_f) = 1$ the final states will coincide, i.e., $\psi'(t_f) = \psi(t_f)$ for a given initial state $\psi'(0) = \psi(0)$. If, in addition, $\dot{\mathcal{U}}(0) = \dot{\mathcal{U}}(t_f) = 0$, then $H(0) = H'(0)$, and $H(t_f) = H'(t_f)$. Let us now list the unitary transformations that provide for the three examples realizable Hamiltonians (Ibáñez et al., 2012a):

- Example 1: Harmonic oscillator expansions,

$$\mathcal{U} = \exp\left(i\frac{m\dot{\omega}}{4\hbar\omega}q^2\right), \quad H' = p^2/(2m) + m\omega'^2q^2/2, \tag{29}$$

where $\omega' = \left[\omega^2 - \frac{3\dot{\omega}^2}{4\omega^2} + \frac{\ddot{\omega}}{2\omega}\right]^{1/2}$.

- Example 2: Harmonic transport,

$$\mathcal{U} = \exp\left(-im\dot{q}_0q/\hbar\right), \quad H' = p^2/(2m) + (q - q_0'(t))^2m\omega_0^2/2, \tag{30}$$

where $q_0' = q_0 + \ddot{q}_0/\omega_0^2$.

- Example 3: Population inversion in a two-level system,

$$\mathcal{U} = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}, \quad H' = \begin{pmatrix} Z_0 - \hbar\dot{\phi}/2 & P \\ P & -Z_0 + \hbar\dot{\phi}/2 \end{pmatrix}, \tag{31}$$

where $\phi = \arctan(\hbar\dot{\Theta}_0/2X_0)$, $0 \le \phi < 2\pi$, and $P = [X_0^2 + (\hbar\dot{\Theta}_0/2)^2]^{1/2}$.

Why do the $\mathcal{U}$s in Eqs. (29-31) have these forms? The answer lies in the symmetry possesed by the Hamiltonian. Transformations of the form $\mathcal{U} = e^{if(t)G_j}$ based on generators $G_j$ of the corresponding Lie algebra produce operators within the algebra and, by suitably manipulating the function $f(t)$ undesired terms may be eliminated.

*Superadiabatic iterations.–* As discussed in Sec. 2.2, Demirplak, Rice, and Berry proposed to add a suitable counterdiabatic (cd) term[5] $H_{cd}^{(0)}$ to the time dependent Hamiltonian $H_0(t)$ so as to follow the adiabatic dynamics of $H_0$. The same $H_{cd}^{(0)}$ also appears naturally when studying the adiabatic approximation of the original system, i.e., the one evolving with $H_0$. This system behaves adiabatically, following the eigenstates of $H_0$, precisely when the counterdiabatic term is negligible.

This is evident in an interaction picture (IP) based on the unitary transformation $A_0(t) = \sum_n |n_0(t)\rangle\langle n_0(0)|$ such that $\psi_1(t) = A_0^\dagger\psi_0$. In this IP, the new Hamiltonian is $H_1(t) = A_0^\dagger(t)(H_0(t) - K_0(t))A_0(t)$ and $K_0(t) = i\hbar\dot{A}_0(t)A_0^\dagger(t)$. If $K_0(t)$ is zero or negligible, $H_1(t)$ becomes diagonal in the basis $\{|n_0(0)\rangle\}$, so that the IP equation is an uncoupled system with solutions

$$|\psi_1(t)\rangle = \sum_n |n_0(0)\rangle e^{-\frac{i}{\hbar}\int_0^t E_n^{(0)}(t')dt'} \langle n_0(0)|\psi_1(0)\rangle. \tag{32}$$

Correspondingly, $|\psi_0(t)\rangle = \sum_n |n_0(t)\rangle e^{-\frac{i}{\hbar}\int_0^t E_n^{(0)}(t')dt'} \langle n_0(0)|\psi_0(0)\rangle$.

The same solution, which, for a non-zero $K_0$, is only approximate, is found exactly by adding to the IP Hamiltonian the counterdiabatic term $A_0^\dagger(t)K_0(t)A_0(t)$. This requires an external intervention and changes the physics of the original system. In the IP the modified Hamiltonian is $H^{(1)} \equiv H_1 + A_0^\dagger(t)K_0(t)A_0(t) = A_0^\dagger(t)H_0(t)A_0(t)$ and in the Schrödinger picture (SP) the modified Hamiltonian is $H_0^{(1)}(t) = H_0(t) + K_0(t)$, so we identify $H_{cd}^{(0)}(t) = K_0(t)$. In other words, a "small" coupling term $K_0$ that makes the adiabatic approximation a good one also implies a small counterdiabatic manipulation. However, irrespective of the size of $K_0$, $H_0^{(1)}(t)$ provides a shortcut to slow adiabatic following because it keeps the populations in the instantaneous basis of $H_0$ invariant, in

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[5] This is the $H_{cd}$ term of Sec. 2.2. The superscript [(0)] is added now to distinguish it from higher order cd terms introduced below.

particular at the final time $t_f$.

Looking for generalized adiabatic approximations, Garrido (1964), Berry (1987) or Deschamps et al. (2008) have investigated further iterative interaction pictures and the corresponding approximations. The idea is best understood by working out explicitly the next iteration: one starts with $i\hbar\partial_t\psi_1(t) = H_1\psi_1(t)$ and diagonalizes $H_1(t)$ to produce its eigenbasis $\{|n_1(t)\rangle\}$. A unitary operator $A_1 = \sum_n |n_1(t)\rangle\langle n_1(0)|$ plays now the same role as $A_0$ in the previous IP. It defines a new IP wave function $\psi_2(t) = A_1^\dagger(t)\psi_1$ that satisfies $i\hbar\partial_t\psi_2(t) = H_2\psi_2(t)$, where $H_2(t) = A_1^\dagger(t)(H_1(t) - K_1(t))A_1(t)$ and $K_1 = i\hbar\dot{A}_1 A_1^\dagger$. If $K_1$ is zero or "small" enough, i.e. if a (first order) superadiabatic approximation is valid, the dynamics would be uncoupled in the new interaction picture, namely,

$$|\psi_2(t)\rangle = \sum_n |n_1(0)\rangle e^{-\frac{i}{\hbar}\int_0^t E_n^{(1)}(t')dt'}\langle n_1(0)|\psi_2(0)\rangle. \tag{33}$$

We may get the same result by changing the physics and adding $A_1^\dagger(t)K_1(t)A_1(t)$ to $H_2$ (Demirplak and Rice, 2008; Ibáñez et al., 2012a). In the SP the added interaction becomes a first order counterdiabatic term $H_{cd}^{(1)} = A_0 K_1 A_0^\dagger$. Transforming back to the SP and using $A_j(0) = 1$ the state (33) becomes

$$|\psi_0(t)\rangle = \sum_n \sum_m |m_0(t)\rangle\langle m_0(0)|n_1(t)\rangle e^{-\frac{i}{\hbar}\int_0^t E_n^{(1)}(t')dt'}\langle n_1(0)|\psi_0(0)\rangle. \tag{34}$$

Quite generally the populations of the final state in the adiabatic basis $\{|n_0(t_f)\rangle\}$ will be different from the ones of the adiabatic process, unless $|n_0(0)\rangle = |n_1(t_f)\rangle$ and $|n_1(0)\rangle = |n_0(0)\rangle$, up to phase factors. The first condition is satisfied if $K_0(t_f) = 0$ and the second one if $K_0(0) = 0$. Then the superadiabatic process will actually lead to the same final populations as an adiabatic one, possibly with different phases for the individual components. Similarly, the first-order counterdiabatic term $H_{cd}^{(1)}$ would provide a shortcut with $H_0^{(2)} = H_0 + H_{cd}^{(1)}$ in the SP, different from the one carried out by $H_0^{(1)}$. Moreover, if $K_1(0) = K_1(t_f) = 0$, then $H_0^{(2)} = H_0$, at $t = 0, t_f$. Further iterations define higher order superadiabatic frames. Is there any advantage in using one or another counterdiabatic scheme? There are two reasons that could make higher order schemes attractive in practice: one is that the structure of the $H_{cd}^{(j)}$ may change with $j$. For example, for a two-level population inversion $H_{cd}^{(0)} = \hbar(\dot{\Theta}_0/2)\sigma_y$, whereas $H_{cd}^{(1)} = \hbar(\dot{\Theta}_1/2)(\cos\Theta_0\sigma_x - \sin\Theta_0\sigma_z)$, where $\Theta_1$ is the polar angle corresponding to the the Cartesian components of $H_1 = X_1\sigma_x + Y_1\sigma_y + Z_1\sigma_z$ (Ibáñez et al., 2012a). The second reason is that, for a fixed process time, the cd-terms are smaller in norm as $j$ increases, up to a value in which they begin to grow, see e.g. Deschamps et al. (2008). One should pay attention though not only to the size of the cd-terms but also to the feasibility of the boundary conditions at the time edges to really generate shortcuts in this manner.

Optimal control theory (OCT) is a vast field covering many techniques and applications. As for STA, fast expansions (Salamon et al., 2009), wavepacket splitting (Hohenester et al., 2007; Grond et al., 2009a,b), transport (Murphy et al., 2009) and many-body state preparation (Rahmani and Chamon, 2011) have been addressed with different OCT approaches. The combination of OCT techniques with invariant-based engineering STA is particularly fruitful since the later provides by construction families of protocols that achieve a perfect fidelity or vanishing final excitation, whereas OCT may help to select among the many possible protocols the ones that optimize some physically relevant variable (Stefanatos et al., 2010, 2011; Chen et al., 2011b; Stefanatos and Li, 2012). In this context the theory used so far is the maximum principle of Pontryagin (1962). For a dynamical system $\dot{x} = \mathbf{f}(\mathbf{x}(t), u)$, where $\mathbf{x}$ is the state vector and $u$ the scalar control, in order to minimize the cost function $J(u) = \int_0^{t_f} g(\mathbf{x}(t), u)dt$, the principle states that the coordinates of the extremal vector $\mathbf{x}(t)$ and of the corresponding adjoint state $\mathbf{p}(t)$ formed by Lagrange multipliers, fulfill Hamilton's equations for a control Hamiltonian $H_c = p_0 g(\mathbf{x}(t), u) + \mathbf{p}^T \cdot \mathbf{f}(\mathbf{x}(t), u)$. For almost all times during the process $H_c$ attains its maximum at $u = u(t)$ and $H_c = c$, where $c$ is constant. We shall discuss specific applications in Secs. 3 and 4.

## 3   Expansions of trapped particles

Performing fast expansions of trapped cold atoms without losing or exciting them is important for many applications: for example to reduce velocity dispersion and collisional shifts in spectroscopy and atomic clocks, decrease the temperature, adjust the density to avoid three body losses, facilitate temperature and density measurements, or to change the size of the cloud for further manipulations. Of course trap compressions are also quite common.

For harmonic traps we may address expansion or compression processes with the quadratic-in-$p$ invariants theory by setting $q_c = U = F = 0$ in Eq. (7). This means that Eq. (9) does not play any role and the important auxiliary equation is the "Ermakov equation" (8). The physical meaning of $\rho$ is determined by its proportionality to the standard deviation of the position of the "expanding (or contracting) modes" $e^{i\alpha_n}\phi_n$.

Here we shall discuss the expansion from $\omega(0) = \omega_0$ to $\omega(t_f) = \omega_f$ (Chen et al., 2010b). Choosing

$$\rho(0) = 1, \quad \dot{\rho}(0) = 0, \tag{35}$$

$H(0)$ and $I(0)$ commute. They actually become equal, and have common eigen-

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functions. Consistent with the Ermakov equation, $\ddot{\rho}(0) = 0$ holds as well for a continuous frequency. At $t_f$ we impose[6]

$$\rho(t_f) = \gamma = (\omega_0/\omega_f)^{1/2}, \quad \dot{\rho}(t_f) = 0, \quad \ddot{\rho}(t_f) = 0. \tag{36}$$

In this manner the expanding mode is an instantaneous eigenvector of $H$ at $t = 0$ and $t_f$, regardless of the exact form of $\rho(t)$. To fix $\rho(t)$, one chooses a functional form to interpolate between these two times, flexible enough to satisfy the boundary conditions. For a simple polynomial ansatz $\rho(t) = 6\,(\gamma - 1)\,s^5 - 15\,(\gamma - 1)\,s^4 + 10\,(\gamma - 1)\,s^3 + 1$ (Palao et al., 1998), where $s = t/t_f$.

The next step is to solve for $\omega(t)$ in Eq. (8). This procedure poses no fundamental lower limit to $t_f$, which could be in principle arbitrarily small. There are nevertheless practical limitations and/or prices to pay. For short enough $t_f$, $\omega(t)$ may become purely imaginary at some $t$ (Chen et al., 2010b) and the potential becomes a parabolic repeller. Another difficulty is that the transient energy required may be too high, as discussed in Chen and Muga (2010) and in the following subsection. Since actual traps are only approximately harmonic, large transient energies will imply perturbing effects of anharmonicities and thus undesired excitations of the final state, or even atom losses.

### 3.1 Transient Energy Excitation

Knowing the transient excitation energy is also important to quantify the principle of unattainability of zero temperature, first enunciated by Nernst. This principle is usually formulated as the impossibility to reduce the temperature of any system to the absolute zero in a finite number of operations, and identified with the third law of thermodynamics. Kosloff and coworkers in (Salamon et al., 2009) have restated the unattainability principle in quantum refrigerators as the vanishing of the cooling rate when the temperature of the cold bath approaches zero, and quantify it by the scaling law that relates cooling rate and cold bath temperature. We shall examine here the consequences of the transient energy excitation on the unattainability principle in two ways: for a single, isolated expansion, and considering the expansion as one of the branches of a quantum refrigerator cycle (Chen and Muga, 2010).

A lower bound $\mathcal{B}_n$ for the time-averaged energy of the n-th expanding mode $\overline{E_n}$ (time averages from 0 to $t_f$ will be denoted by a bar) is found by applying

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[6] If $\ddot{\rho}(t_f) \neq 0$ the final frequency would not be $\omega_f$ but $\omega(t_f) = [\omega_f^2 - \ddot{\rho}/\gamma]^{1/2}$. If discontinuities are allowed and the frequency is changed abruptly from $\omega(t_f)$ to $\omega_f$ the excitations will also be avoided, at least in principle. A similar discontinuity is possible at $t = 0$ if $\ddot{\rho}(0) \neq 0$ and the frequency jumps abruptly from $\omega_0$ to $\omega(0) = [\omega_0^2 - \ddot{\rho}(0)]^{1/2}$.

calculus of variations (Chen and Muga, 2010), so that $\overline{E_n} \geq \mathcal{B}_n$. If the final frequency $\omega_f$ is small enough to satisfy $t_f \ll 1/\sqrt{\omega_0 \omega_f}$, and $\gamma \gg 1$, the lower bound has the asymptotic form $\mathcal{B}_n \approx (2n+1)\hbar/(2\omega_f t_f^2)$. A consequence is that $t_f \geq \sqrt{(2n+1)\hbar/(2\omega_f \overline{E_n})}$. When $\overline{E_n}$ is limited, because of anharmonicities or a finite trap depth, the scaling is fundamentally the same as the one found for bang-bang methods with real frequencies (Salamon et al., 2009), and leads to a cooling rate $R \propto T_c^{3/2}$ in an inverse quantum Otto cycle (the proportionality factor may be improved by increasing the allowed $\overline{E_n}$). This dependence had been previously conjectured to be a universal one characterizing the unattainability principle for any cooling cycle (Rezek et al., 2009). The results in Chen and Muga (2010) provide strong support for the validity of this conjecture within the set of processes defined by ordinary harmonic oscillators with time-dependent frequencies. In (Hoffmann et al., 2011) a faster rate $\sim -T_c/\log T_c$ is found with optimal control techniques for bounded trap frequencies, allowed to become imaginary. There is no contradiction with the previous scaling since bounding the trap frequencies does not bound the system energy. In other words, achieving such fast cooling is not possible if the energy cannot become arbitrarily large.

Independently of the participation of the harmonic trap expansion as a branch in a refrigerator cycle, we may apply the previous analysis also to a single expansion, assuming that the initial and final states are canonical density operators characterized by temperatures $T_0$ and $T_f$. These are related by $T_f = (\omega_f/\omega_0)T_0$ for a population-preserving process. In a harmonic potential expansion, the unattainability of a zero temperature can be thus reformulated as follows: The transient excitation energy becomes infinite for any population-preserving and finite-time process when the final temperature is zero (which requires $\omega_f = 0$). The excitation energy has to be provided by an external device, so a fundamental obstruction to reach $T_f = 0$ in a finite time, is the need for a source of infinite power (Chen and Muga, 2010).

The standard deviation of the energy was also studied numerically (Chen and Muga, 2010). There it was found that the dominant dependences of the time averages scale with $\omega_f$ and $t_f$ in the same way as the average energy. These dependences are different from the ones in the Anadan and Aharonov (1990) (AA) relation $\overline{\Delta H}\, t_f \geq \frac{h}{4}$, where $\overline{\Delta H} = \int_0^{t_f} \Delta H(t) dt/t_f$.

### 3.2 Three Dimensional Effects

The previous discussion is limited to one dimension (1D) but actual traps are three-dimensional and at most effectively 1D. Torrontegui et al. (2012c) worked out the theory and performed numerical simulations of fast expansions of cold atoms in a three-dimensional Gaussian-beam optical trap. Three

different methods to avoid final motional excitation were compared: inverse engineering using Lewis-Riesenfeld invariants, which provides the best overall performance, a bang-bang approach with one intermediate frequency, and a "fast adiabatic approach" [7].

The optical trap considered in Torrontegui et al. (2012c) is formed by a laser, red detuned with respect to an atomic transition, and is characterized in the harmonic approximation by longitudinal and radial frequencies. To fourth order in the coordinates the effective potential includes anharmonic terms and radial-longitudinal coupling terms. While magnetic traps allow for an independent control of longitudinal and radial frequencies (Schaff et al., 2010, 2011a,b), this is not the case for a simple laser trap. In Torrontegui et al. (2012c) it was assumed that the time-dependence of the longitudinal frequency is engineered to avoid final excitations with a simple 1D harmonic theory. The main conclusion of the study is that the transitionless expansions in optical traps are feasible under realistic conditions. For the inverse engineering method, the main perturbation is due to the possible adiabaticity failure in the radial direction, which can be suppressed or mitigated by increasing the laser waist. This waist increase would also reduce smaller perturbing effects due to longitudinal anharmonicity or radial-longitudinal coupling. The simple bang-bang approach fails because the time for the radial expansion is badly mismatched with respect to the ideal time, and the fast adiabatic method fails for short expansion times as a result of longitudinal excitations. Complications such as perturbations due to different noise types, and consideration of condensates, gravity effects, or the transient realization of imaginary trap frequencies are still open questions. Other extensions of Torrontegui et al. (2012c) could involve the addition of a second laser for further control of the potential shape, or alternative trap shapes. Optical traps based on Bessel laser beams, for example, may be useful to decouple longitudinal and radial motions.

*3.3 Bose-Einstein Condensates*

In this section we shall discuss the possibility of realizing STA in a harmonically trapped Bose-Einstein condensate using a scaling ansatz. A mean-field description of this state of matter is based on the time-dependent Gross-Pitaevskii equation (GPE) (Dalfovo et al., 1999),

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[7] The adiabaticity condition for the harmonic oscillator is $|\sqrt{2}\dot{\omega}/(8\omega^2)| \ll 1$. An efficient, but still adiabatic, strategy by Chen et al. (2010b) is to distribute $\dot{\omega}/\omega^2$ uniformly along the trajectory, i.e., $\dot{\omega}/\omega^2 = c$, $c$ being constant. Solving this differential equation and imposing $\omega_f = \omega(t_f)$ we get $\omega(t) = \omega_0/[1 - (\omega_f - \omega_0)t/(t_f\omega_f)]$. This may be enough for some applications. This function was successfully applied in Bowler et al. (2012).

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\Delta + \frac{1}{2}m\omega^2(t)\mathbf{x}^2 + g_D|\Psi(\mathbf{x},t)|^2\right]\Psi(\mathbf{x},t). \tag{37}$$

Here, $\Delta$ is the $D$-dimensional Laplacian operator and $g_D$ is the $D$-dimensional coupling constant. For a three-dimensional cloud, using the normalization $\int|\Psi(\mathbf{x},t)|^2d\mathbf{x} = 1$, $g_3 = \frac{4\pi\hbar^2 Na}{m}$, for a condensate of a number of atoms $N$ of mass $m$, interacting with each other through a contact Fermi-Huang pseudopotential parameterized by a $s$-wave scattering length $a$. In $D = 1,2$ the corresponding expression for $g_D$ can be obtained by a dimensional reduction of the 3D GPE (Salasnich et al., 2002). As a mean-field theory the GPE overestimates the phase coherence of real Bose-Einstein condensates. The presence of phase fluctuations generally induces a breakdown of the dynamical self-similar scaling law that governs the dynamics of the expanding cloud and the formation of density ripples. The conditions for quantum phase fluctuations to be negligible for STA were discussed in del Campo (2011a). In the following we shall ignore phase fluctuations. The results of this section will be generalized to strongly correlated gases in Sec. 3.4, including as a particular case, the microscopic model of ultracold bosons interacting through s-wave scattering.

STA in the mean-field regime were designed in Muga et al. (2009) based on the classic results by Castin and Dum (1996), and Kagan, Surkov and Shlyapnikov (1996), who found the exact dynamics of the condensate wavefunction under a time-modulation of the harmonic trap frequency. Consider a condensate wavefunction $\Psi(\mathbf{x},t = 0)$, a solution of the time-independent GPE with chemical potential $\mu$ in a harmonic trap of frequency $\omega_0$, i.e., $(-\frac{\hbar^2}{2m}\Delta + \frac{1}{2}m\omega_0^2\mathbf{x}^2 + g_D|\Psi(\mathbf{x},t = 0)|^2 - \mu)\Psi(\mathbf{x},t = 0) = 0$. Under a modulation of the trap frequency $\omega(t)$ the scaling ansatz

$$\Psi(\mathbf{x},t) = \frac{1}{\rho^{\frac{D}{2}}}\exp\left[i\frac{m|\mathbf{x}|^2}{2\hbar}\frac{\dot{\rho}}{\rho} - i\frac{\mu\tau(t)}{\hbar}\right]\Psi\left(\frac{\mathbf{x}}{\rho},t = 0\right) \tag{38}$$

is an exact solution of the time-dependent Gross-Pitaevskii equation provided that

$$\ddot{\rho} + \omega(t)^2\rho = \frac{\omega_0^2}{\rho^3}, \qquad g_D(t) = \frac{g_D(t = 0)}{\rho^{2-D}}, \qquad \tau(t) = \int_0^t\frac{dt'}{\rho^2}. \tag{39}$$

It follows that the scaling factor $\rho$ must be a solution of the Ermakov equation, precisely as in the single-particle harmonic oscillator case. This paves the way to engineer a shortcut to an adiabatic expansion or compression from the initial state $\Psi(\mathbf{x},t = 0)$ to a target state $\Psi(\mathbf{x},t_f) = \Psi(\mathbf{x}/\rho,t = 0)/\rho^{\frac{D}{2}}$ by designing the trajectory $\rho(t)$. The modulation of the coupling constant required in $D = 1,3$ can be implemented with the aid of a Feshbach resonance (Muga et al., 2009), or, in $D = 1$, by a modulation of the transverse

confinement (Staliunas et al., 2004; Engels et al., 2007; del Campo, 2011a). The $D = 2$ requires no tuning in time of the coupling constant as a result of the Pitaevskii-Rosch symmetry (Pitaevskii and Rosch, 1997). It has recently been suggested that this symmetry is broken upon quantization, constituting an instance of a quantum anomaly in ultracold gases (Olshanii et al., 2010). To date no experiment has provided evidence in favor of this observation. We point out that observing a breakdown of shortcuts to expansions of 2D BEC clouds would help to verify this quantum-mechanical symmetry breaking.

An important simplification occurs in the Thomas-Fermi regime, where the mean-field energy dominates over the kinetic part. Assuming the validity of this regime along the dynamics, the scaling ansatz (38) becomes exact as long as the following consistency equations are satisfied,

$$\ddot{\rho} + \omega(t)^2 \rho = \frac{\omega_0^2}{\rho^{D+1}}, \qquad g_D(t) = g_D(t = 0), \qquad \tau(t) = \int_0^t \frac{dt'}{\rho^D}. \tag{40}$$

Hence, in the Thomas-Fermi regime, it is possible to engineer a shortcut exactly, while keeping the coupling strength $g_D$ constant (Muga et al., 2009). Optimal control theory has been recently applied in this regime to find optimal protocols with a restriction on the allowed frequencies Stefanatos and Li (2012).

*Dimensional reduction and modulation of the non-linear interactions.*– For low dimensional BECs, tightly confined in one or two directions, an effective tuning of the coupling constant can be achieved by modulating the trapping potential along the tightly confined axis, see e.g. Staliunas et al. (2004), a proposal experimentally explored in Engels et al. (2007). In a nutshell, the tightly confined degrees of freedom decoupled from the weakly confined ones, are governed to a good approximation by a non-interacting Hamiltonian. It is then possible to perform a dimensional reduction of the 3D GPE, and derive a lower-dimensional version for the weakly confined degrees of freedom, where the effective coupling constant inherits a dependence of the width of the transverse modes which have been integrated out. Adiabatically tuning the transverse confinement leads to a controlled tuning of the effective coupling constant. A faster-than-adiabatic modulation can be engineered by implementing a shortcut in the transverse degree of freedom. Consider the 3D mean-field description

$$i\hbar \frac{\partial \Psi(\mathbf{x}, t)}{\partial t} = \Big[ -\frac{\hbar^2}{2m}\Delta + V^{\text{ex}}(\mathbf{x}, t) + g_3|\Psi(\mathbf{x}, t)|^2 \Big]\Psi(\mathbf{x}, t), \tag{41}$$

with $V^{\text{ex}}(\mathbf{x}, t) = \frac{m}{2}[\omega_x(t)^2 x^2 + \omega_y(t)^2 y^2 + \omega_z(t)^2 z^2]$. For tight transverse confinement ($\omega_x \sim \omega_y \gg \omega_z$ and $N|a|\sqrt{m\omega_z/\hbar} \ll 1$), the transverse excitations are

frozen. The transverse mode can be approximated by the single-particle harmonic oscillator ground state $\Phi_0(x, y, t)$, so that the wavefunction factorizes $\Psi(\mathbf{x}, t) = \Phi_0(x, y, t)\psi(z, t)$. Integrating out the transverse modes, and up to a time-dependent constant which can be gauged away, one obtains the reduced GPE

$$i\hbar\frac{\partial\psi(z,t)}{\partial t} = \Big[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + V^{\text{ex}}(z) + g_1(t)|\psi(z,t)|^2\Big]\psi(z,t), \tag{42}$$

with the effective coupling $g_1(t) = g_3 \iint dx dy |\Phi_0(x, y, t)|^4$. A general trajectory $g_1(t)$ can be implemented by modifying the frequency $\omega_\perp(t)$ of the transverse confinement according to

$$\omega_\perp^2(t) = \omega_\perp^2(0)\left[\frac{g_1(t)}{g_1(0)}\right]^2 + \frac{1}{2}\frac{\ddot{g}_1(t)}{g_1(t)} - \frac{3}{4}\left[\frac{\dot{g}_1(t)}{g_1(t)}\right]^2 \tag{43}$$

in quasi-1D atomic clouds (del Campo, 2011a). The first term in the RHS corresponds to the adiabatic tuning discussed in Staliunas et al. (2004); Engels et al. (2007) while the remaining terms are associated with the STA dynamics in the transverse modes. A similar analysis applies to the control of the effective coupling constant in a pancake condensate, in the x-y plane, under tight confinement along the $z$-direction (del Campo, 2011a).

We note that this technique is restricted to tune the amplitude of the coupling constant, at variance with alternative techniques based on Feschbach or confinement-induced resonances which can change both the amplitude and character of the interactions, e.g., from attractive to repulsive (Bloch et al., 2008).

### 3.4 Strongly Correlated Gases

The preceding sections were focused on single-particle systems and a mean-field description of Bose-Einstein condensates. We have seen that the inversion of scaling laws is a powerful technique to design STA in those processes where the dynamics is self-similar, e.g. expansions, or transport. In the following we focus on the engineering of STA in strongly correlated quantum fluids of relevance to ultracold gases experiments. We shall consider a fairly general model in dimension $D$ consisting of $N$ indistinguishable particles with coordinates $\mathbf{x}_i \in \mathbb{R}^D$, trapped in a time-dependent isotropic harmonic potential of frequency $\omega(t)$ and interacting with each other through a two-body potential $V(\mathbf{x}_i - \mathbf{x}_j)$. The many-body Hamiltonian describing this system reads (del Campo, 2011b)

$$\mathcal{H} = \sum_{i=1}^{N} \left[ -\frac{\hbar^2}{2m}\Delta_i + \frac{1}{2}m\omega^2(t)\mathbf{x}_i^2 \right] + \epsilon \sum_{i<j} \mathrm{V}(\mathbf{x}_i - \mathbf{x}_j), \tag{44}$$

where $\Delta_i$ is the $D$-dimensional Laplacian operator for the $\mathbf{x}_i$ variable, and $\epsilon = \epsilon(t)$ is a dimensionless time-dependent coupling strength satisfying $\epsilon(0) = 1$. We shall further assume that $\mathrm{V}(\lambda\mathbf{x}) = \lambda^{-\alpha}\mathrm{V}(\mathbf{x})$ under scaling of the co-ordinates. Specific realizations of this model include the Calogero-Sutherland model (Sutherland, 1998), the Tonks-Girardeau gas (Öhberg and Santos, 2002; Minguzzi and Gangardt, 2005), Lieb-Liniger gas (Bujan et al., 2008), Bose-Einstein condensates (BEC) (Castin and Dum, 1996; Kagan, Surkov and Shlyapnikov, 1996; Muga et al., 2009), including dipolar interactions (O'Dell et al., 2004), and more general many-body quantum systems (Gritsev et al., 2010). For simplicity, we leave out other cases to which similar techniques can be applied, such as strongly interacting mixtures (Minguzzi and Girardeau, 2007) or systems with internal structure (Mousavi et al., 2007; Deuretzbacher et al., 2008).

Let us now consider an equilibrium state $\Phi$ of the system (44) at $t = 0$ with chemical potential $\mu$. For compactness we shall use the notation $\mathbf{x}_{j:k} \equiv \{x_j, x_{j+1}, \ldots, x_{k-1}, x_k\}$. It is possible to find a self-similar scaling solution of the form

$$\Phi\left(\mathbf{x}_{1:N}, t\right) = \frac{1}{\rho^{D/2}} \exp\left[ i\sum_{i=1}^{N} \frac{m\mathbf{x}_i^2 \dot{\rho}}{2\rho\hbar} - i\mu\tau(t)/\hbar \right] \Phi\left(\frac{\mathbf{x}_{1:N}}{\rho}, t=0\right), \tag{45}$$

where $\tau(t) = \int_0^t dt'/\rho^2(t')$, whenever the scaling factor $\rho = \rho(t)$ is the solution of the Ermakov differential equation, $\ddot{\rho} + \omega^2(t)\rho = \omega_0^2/\rho^3$, with $\omega_0 = \omega(0)$, satisfying the boundary conditions $\rho(0) = 1$ and $\dot{\rho}(0) = 0$. This is the same consistency equation that arises in the context of the single-particle time-dependent harmonic oscillator.

Scaling laws greatly simplify the dynamics of quantum correlations. Let us consider the time-evolution of the $n$-particle reduced density matrix

$$g_n(\mathbf{x}_{1:n}; \mathbf{x}'_{1:n}; t) = \frac{N!}{(N-n)!} \int \prod_{i=n+1}^{N} dx_i \Phi^*(\mathbf{x}_{1:N}; t)\Phi(\mathbf{x}'_{1:n}, \mathbf{x}_{n+1:N}; t). \tag{46}$$

Provided the scaling law holds, its time evolution reads

$$g_n(\mathbf{x}_{1:n}; \mathbf{x}'_{1:n}; t) = \rho^{-nD} g_n\left(\frac{\mathbf{x}_{1:n}}{\rho}; \frac{\mathbf{x}'_{1:n}}{\rho}; 0\right) \exp\left(-\frac{i}{\rho}\frac{\dot{\rho}}{\omega_0}\frac{\sum_{i=1}^{n}(\mathbf{x}_i^2 - \mathbf{x}'^2_i)}{2l_0^2}\right), \tag{47}$$

where $l_0 = \sqrt{\hbar/m\omega_0}$.

Local quantum correlations depend exclusively on the diagonal elements of $g_n(\mathbf{x}_{1:n}; \mathbf{x}'_{1:n}; t)$ and manifest directly the self-similar dynamics. For instance, the time evolution of the density profile $n(\mathbf{x}) = g_1(\mathbf{x}; \mathbf{x})$ reads $n(\mathbf{x}, t) = \rho^{-nD} n\left(\frac{\mathbf{x}}{\rho}, t = 0\right)$.

The dynamics of non-local correlations is more involved due to the presence of the oscillatory phase. As an example, the evolution of the one-body reduced density matrix (OBRDM) under self-similar dynamics (Minguzzi and Gangardt, 2005; Gritsev et al., 2010),

$$g_1(\mathbf{x}, \mathbf{y}; t) = \frac{1}{\rho^D} g_1\left(\frac{\mathbf{x}}{\rho}, \frac{\mathbf{y}}{\rho}; 0\right) \exp\left(-\frac{i}{\rho}\frac{\dot{\rho}}{\omega_0}\frac{\mathbf{x}^2 - \mathbf{y}^2}{2l_0^2}\right), \qquad (48)$$

induces a non-self-similar evolution of the momentum distribution, its Fourier transform $n(\mathbf{k}, t) = \int d\mathbf{x} d\mathbf{y} \, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} g_1(\mathbf{x}, \mathbf{y}; t)$. It is expected that the oscillatory phases distort quantum correlations. The case of a free expansion, where the frequency modulation in terms of the Heaviside function $\Theta(t)$ reads $\omega(t) = \omega_0 \Theta(-t)$, has received much attention. The solution to the Ermakov equation for the scaling factor is $\rho(t) = \sqrt{1 + \omega_0^2 t^2}$ and for $t \gg \omega_0^{-1}$, $\rho(t) \sim \omega_0 t$, $\dot{\rho} = \omega_0$. Using the method of the stationary phase, it follows that

$$n(\mathbf{k}, t) \sim |2\pi\omega_0 l_0^2/\dot{\rho}|^D g_1(\omega_0 \mathbf{k} l_0^2/\dot{\rho}, \omega_0 \mathbf{k} l_0^2/\dot{\rho}), \qquad (49)$$

i.e., the asymptotic momentum distribution is mapped to the scaled density profile of the initial state (Jukić et al., 2008; Oezer et al., 2009; Gritsev et al., 2010). As a result, all information of the off-diagonal elements of the OBRDM is lost. Similar effects result in an expansion in finite-time $t_f \sim \omega_0^{-1}$ and signal the breakdown of adiabaticity. Excitations manifest as well in local correlation functions, e.g. excitation of the breathing mode of the cloud.

In the adiabatic limit ($\tau \gg \omega_0^{-1}$), the time-variation of the the scaling factor vanishes $\dot{\rho}(t) \approx 0$, resulting in the adiabatic trajectory $\rho(t) = \sqrt{\omega_0/\omega(t)}$. At all times the time-evolution of the OBRDM and the momentum distribution can be related by a scaling transformation of their form at $t = 0$,

$$g_1(\mathbf{x}, \mathbf{y}; t) = \frac{1}{\rho^D(t)} g_1\left(\frac{\mathbf{x}}{\rho(t)}, \frac{\mathbf{y}}{\rho(t)}; 0\right), \qquad n(\mathbf{k}, t) = \rho^D(t) n(\rho(t)\mathbf{k}, 0). \quad (50)$$

These expressions can be applied for expansions ($\rho(t) > 1$) and compressions ($\rho(t) < 1$), and generally still require tuning the interaction coupling strength. Nonetheless, the required adiabatic time scale can be exceedingly long and we next tackle the problem of achieving a final scaled state in a predetermined expansion time $t_f$. The upshot of the frictionless dynamics is that quantum correlations at the end of the quench ($t = t_f$, and only then) are those of the initial state scaled by a factor $\rho(t_f) = \gamma$ (del Campo, 2011b). In particular,

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$$g_1(\mathbf{x}, \mathbf{y}; t_f) = \frac{1}{\gamma^D} g_1 \left( \frac{\mathbf{x}}{\gamma}, \frac{\mathbf{y}}{\gamma}; 0 \right), \qquad n(\mathbf{k}, t_f) = \gamma^D n(\gamma \mathbf{k}, 0). \tag{51}$$

Similar expressions hold for higher-order correlations, i.e. $g_n(\mathbf{x}_{1:n}, \mathbf{y}_{1:n}; t_f) = \gamma^{nD} g_n (\mathbf{x}_{1:n}/\gamma, \mathbf{y}_{1:n}/\gamma; 0)$. Moreover, as long as the initial state is an equilibrium state in the initial trap, so it is the state at $t_f$ with respect to the final trap, preventing any non-trivial dynamics after the quench, for $t > t_f$ if $\omega(t > t_f) = \omega_f$. Nonetheless, at intermediate times $t \in [0, t_f)$ the momentum distribution exhibits a rich non-equilibrium dynamics, and can show for instance, evolution towards the scaled density profile of the initial state.

We close this section with two comments. First, the applicability of STA based on inversion of scaling laws is not restricted to fermionic or bosonic systems, but can be applied as well to anyonic quantum fluids for which dynamical scaling laws are known (del Campo, 2008). Systems with quantum statistics smoothly extrapolating between bosons and fermions might be realized in the laboratory following Keilmann et al. (2011). Second, the possibility of scaling up the system while preserving quantum correlations constitutes a new type of microscopy of quantum correlations in quantum fluids (del Campo, 2011b; del Campo and Boshier, 2012). It is as well of interest to design new protocols to reconstruct the initial quantum state of the system from the time-evolution of its density profile (Bertrand and Betrand, 1987; Leonhardt and Schneider, 1997), a tomographic technique demonstrated experimentally in Kurtsiefer et al. (1997), and applicable to many-body systems (del Campo et al., 2008b).

*Scaling laws in other trapping potentials.*– Scaling laws for many-body systems can be found for more general types of confinements. Among them, homogeneous potentials are of particular interest, since they simplify the correspondence between ultracold atom experiments and condensed matter theory. The early experimental implementations of the paradigmatic particle in a box aimed at the creation of optical billiards for ultracold gases (Milner et al., 2001; Friedman et al., 2001). Trapping of a BEC in an all-optical box was reported in Meyrath et al. (2005) and analogous traps have been created in atom chips (van Es et al., 2010). For the purpose of implementing STA, the dynamical optical dipole potential may be realized using the highly versatile "painting technique", which creates a smooth and robust time-averaged potential with a rapidly-moving laser beam (Henderson et al., 2009), or alternatively, by spatial light modulators (Boyer et al., 2006).

The breakdown of adiabaticity in a time-dependent homogeneous potential leads to quantum transients related to the diffraction in time (DIT) effect, see del Campo et al. (2009) for a review. A sharply localised matter-wave in a region of space, after sudden removal of the confinement, exhibits during free evolution density ripples. The earliest example discussed by Moshinsky (1952) the free evolution of a truncated cut-off plane wave, exhibits an oscil-

latory pattern with the same functional form than the diffraction pattern of a classical light beam from a semi-infinite plane. The phenomenon is ubiquitous in matter-wave dynamics induced by a quench, and in particular, it arises in time-dependent box potentials in one (Gerasimov and Kazarnovskii, 1976; Godoy, 2002; del Campo and Muga, 2006), two and three (Godoy, 2003) dimensions. The effect manifests as well in strongly interacting gases such as ultracold bosons in the Tonks-Girardeau regime (del Campo and Muga, 2006; del Campo, 2008). Moreover, when the piston walls move at a finite speed $v$, the adiabatic limit is not approached monotonically as $v \to 0$. It was shown in del Campo et al. (2008a); Mousavi (2012a,b) that an enhancement of DIT occurs when the walls move with the dominant velocity component of the initial confined state, due to a constructive interference between the expanding and reflected components from the walls. When the reflections from the confinement walls dominate, the non-adiabatic dynamics in time-dependent homogeneous potentials lead to Talbot oscillations and weave a quantum carpet in the time evolution of the density profile (Friesch et al., 2000; Ruostekoski et al., 2001). Suppression of these excitations is dictated by the adiabatic theorem both in the non-interacting (Berry and Klein, 1984; Dodonov et al., 1993; Chen et al., 2009; Mostafazadeh, 2001) and mean-field regime (Band et al., 2002). Further, for non-interacting systems one can prove that no shortcut based on invariants or scaling exists in time-dependent homogeneous potentials. At the single-particle level, this follows from the fact that the family of trajectories for the width $\xi(t)$ of a box-like potential for which a dynamical invariant exist (Berry and Klein, 1984), takes the form $\xi(t) = [at^2 + bt + c]^{\frac{1}{2}}$, which is incompatible with the boundary conditions required to reduce a time-evolving scaling solution to the initial and target states. For many-body quantum fluids, the same result is derived from the consistency equations for self-similar dynamics to occur. To find a shortcut in this scenario one has to relax the condition on the confinement and allow for a inhomogeneous auxiliary harmonic potential of the form (del Campo and Boshier, 2012)

$$U^{\mathrm{aux}}(\mathbf{x}, t) = -\frac{1}{2} m \frac{\ddot{\xi}(t)}{\xi(t)} |\mathbf{x}|^2, \tag{52}$$

where $\mathbf{x} \in \mathbb{R}^D$, $|\mathbf{x}| \in [0, \xi(t)]$. For $D = 1$, a box with one stationary wall at $x = 0$ and moving wall at $x = \xi(t)$ is assumed. Cylindrical and spherical symmetry is imposed for $D = 2, 3$ respectively. This auxiliary potential can be implemented by means of a blue-detuned laser (Khaykovich et al., 2002) or direct painting with a rapidly moving laser (Milner et al., 2001; Friedman et al., 2001; Henderson et al., 2009). Thanks to its presence it is possible to find dynamical self-similar solutions to single-particle and many-body Schrödinger equations for time-dependent box-like confinements with a general modulation of the width $\xi(t)$. In particular, consider the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \left[ -\frac{\hbar^2}{2m}\Delta_i + U^{\text{aux}}(\mathbf{x}_i, t) \right] + \epsilon \sum_{i<j} V(\mathbf{x}_i - \mathbf{x}_j), \tag{53}$$

where $\mathbf{x}_i \in \mathbb{R}^D$, $r_i = |\mathbf{x}|_i \in [0, \xi(t)]$, and let us introduce the scaling factor $\rho(t) = \xi(t)/\xi(0)$. If $V(\lambda\mathbf{x}) = \lambda^{-\alpha}V(\mathbf{x})$, $\epsilon(t) = \rho(t)^{\alpha-2}$, in the presence of $U^{\text{aux}}(\mathbf{x}, t)$, the time evolution of an initial eigenstate of the system with chemical potential $\mu$ follows a scaling law in Eq. (45). Given the existence of a scaling law, a many-body shortcut can be engineered by designing the scaling factor as for the simple harmonic oscillator, ensuring that the time-evolving state reduces to the initial and target state at the begining and end of the evolution (del Campo and Boshier, 2012). Naturally, this is possible as well for Bose-Einstein condensates in the mean-field regime, extending the Castin-Dum-Kagan-Surkov-Shlyapnikov scaling ansatz (del Campo and Boshier, 2012).

Along a shortcut to an adiabatic expansion, the auxiliary potential is expulsive in an early stage of the expansion, expelling the atoms from the center and providing the required speed-up. The rapidly expanding cloud is slowed down in a second state of the expansion, when $U^{\text{aux}}(\mathbf{x}, t)$ becomes a trapping potential. The sequence is reversed in a shortcut to an adiabatic compression. In both cases, at $t = t_f$, $U^{\text{aux}}(\mathbf{x}, t)$ vanishes, and the cloud reaches the target state, a stationary state of the final Hamiltonian. As a result, STA provide a variant of the paradigmatic model of a quantum piston (Quan and Jarzynski, 2012).

## 3.5  Experimental Realization

Experiments of fast shortcut expansions have been realized at Nice with magnetic confining of $^{87}$Rb atoms for ultra-cold clouds (Schaff et al., 2010) and condensates in the Tomas-Fermi regime (Schaff et al., 2011a). Compared to the simple expansions treated in (Chen et al., 2010b), gravity introduces and extra linear term in the Hamiltonian and requires a treatment with additional boundary conditions.

For the cold cloud, samples of $N = 10^5$ atoms and temperature $T_0 = 1.63$ $\mu$K were used to keep the time between collisions small $\approx 28$ ms, and the potential effectively harmonic. The initial trap frequencies for $x, y, z$ directions in Hz were $(228.1, 22.2, 235.8)$ and the final ones $(18.1, 7.1, 15.7)$. The results for the fast (35 ms) 15-fold frequency decompression to the trap in the vertical dimension, yielded a residual center-of-mass oscillation of the cloud equivalent to that of a 1.3-s-long linear decompression, a reduction by a factor of 37.

For the condensate, the number of atoms was $N = 1.3 \times 10^5$ and the ini-

tial temperature was $T_0 = 130$ nK (Schaff et al., 2011a). The potential is $U(r,t) = m\omega_\perp^2(t)(x^2 + z^2) + \frac{1}{2}m\omega_\parallel^2(t)y^2 + mgz$. Initial radial $(x, z)$ and axial $(y)$ frequencies were 235.8 Hz and 22.2 Hz, respectively. The experiment performed a 30-ms-long radial decompression of the trap by a factor of 9, yielding a final radial frequency of 26.2 Hz. The axial frequency was reduced by a factor of 3 to a final value 7.4 Hz. Using scaling techniques similar to the ones in Sec. 3.3 it was shown that this decompression is a shortcut for both directions. Residual excitations were attributed to imperfect implementation of $\omega(t)$, anharmonicities, and trap tilting.

## 3.6   Optimal Control

The time-dependent frequency of a harmonic trap expansion based on invariants can be optimized with respect to time or to transient excitation energy, restricting the allowed transient frequencies (Stefanatos et al., 2010, 2011).

Kosloff and coworkers have applied OCT to minimize the expansion time with "frictionless conditions", i.e., taking an initial thermal equilibrium at one temperature into thermal equilibrium at another temperature in a cooling cycle, using real or imaginary bang-bang (piecewise constant or ramped) intermediate trap frequencies, see e.g. Salamon et al. (2009); Hoffmann et al. (2011).

## 3.7   Other Applications

Inverse engineering expansions using invariant theory or scaling laws have been applied in several contexts. For example, Choi, Onofrio and Sundaram (2011) discussed the possibility of achieving deep degeneracy of Fermi gases via sympathetic cooling by changing the trapping frequency of another species (the coolant) to keep constant the Lewis-Riesenfeld invariant. The identified advantages are the maximal heat capacity retained by the coolant due to the conservation of the number of atoms, and the preservation of its phase-space density in the nondegenerate regime where the specific heat retains its Dulong-Petit value. The limits of the approach are set by the transient excitation, that should be kept below some allowed threshold, and by the spreading of the cooling cloud which reduces the spatial overlap with the Fermionic cloud. The method is found to be quite robust with respect to broadband noise in the trapping frequency (Choi et al., 2012).

Li et al. (2011) propose a scheme to cool down a mechanical resonator in a three-mirror cavity optomechanical system. The dynamics of the mechanical resonator and cavities is reduced to that of a time-dependent harmonic oscillator, whose effective frequency can be controlled through the optical driving

fields. A simpler harmonic system is studied in Zhang et al. (2012a), a charged mechanical resonator coupled to electrodes via Coulomb interaction controlled by bias gate voltages.

Yuce (2012) designs, using scaling, fast frictionless expansions of an optical lattice with dynamically variable spacing (accordion lattice). Specifically, he considers the 1D Hamiltonian $H = p^2/(2m) + V(t)\cos(2k_L x/\Lambda(t)) + m\omega^2(t)x^2/2$, where $\Lambda$ is the scale parameter which goes from 1 at $t = 0$ to $c$ at $t_f$ and the parabolic potential only acts during the expansion according to $\omega^2(t) = -\Lambda^{-1}\partial^2\Lambda/\partial t^2$. Decreasing the potential depth as $V(t) = V_0/\Lambda^2(t)$, and making the first and second derivatives of $\Lambda$ vanish at the boundary times guarantee a frictionless expansion. In Ozcakmakli and Yuce (2012) the results are extended to a continuously replenished BEC in a harmonic trap or in an optical lattice.

Lau and James (2012) propose inverse engineering of the trap frequencies based on the Lewis-Riesenfeld invariants as part of the elementary operations necessary to implement a universal bosonic simulator using ions in separate traps. This method would allow to improve the accuracy and speed of conventional laser operations on ions which are limited by the Lamb-Dicke approximation.

Juliá-Díaz et al. (2012) develop a method to produce highly coherent-spin-squeezed many-body states in bosonic Josephson junctions (BJJs). They start from the known mapping of the two-site Bose-Hubbard (BH) Hamiltonian to that of a single effective particle evolving according to a Schrödinger-like equation in Fock space. Since, for repulsive interactions, the effective potential in Fock space is nearly parabolic, the inversion protocols for shortcuts to adiabatic evolution in harmonic potentials may be applied to the many-body BH Hamiltonian. The procedure requires a good control of the time variation of the atom-atom scattering length during the desired period, a possibility now at hand in current experimental setups for internal BJJs.


## 4   Transport


The efficient transport of atoms and ions by moving the confining trap is a necessary fundamental requirement for many applications. These are for example quantum information processing in multiplexed trap arrays (Rowe et al., 2002; Reichle et al., 2006; Bowler et al., 2012) or quantum registers (Miroschnychenko et al., 2008); controlled translation from the production or cooling chamber to the interaction or manipulation zones; control of interaction times and locations, e.g. in cavity QED experiments, quantum gates (Calarco et al., 2000) or metrology (Prestage et al., 1993); and velocity control

to stop (Schmidt et al., 2009; Chen et al., 2009) or launch atoms (Kuhr et al., 2001).

The transport should ideally be lossless, fast and "faithful", i.e. the final state should be equal to the initial one apart from the translation and possibly phase factors. This is compatible with some transient excitation in the instantaneous basis at intermediate times.

Many different experimental approaches have been implemented. Neutral atoms have been transported individually, as thermal atomic clouds, or condensates, using optical or magnetic traps. The magnetic traps can be displaced by moving the coils mechanically, by time-varying currents in a lithographic pattern, or on a conveyor belt with permanent magnets (Lahaye et al., 2006). Optical traps can be used as optical tweezers whose focal point is translated by moving mechanically lenses (Couvert et al., 2008a), and traveling lattices (conveyor belts) can be made with two counterpropagating beams slightly detuned. Mixed magneto-optical approaches are also possible. To transport ions, controlled time dependent voltages have been used in linear-trap based frequency standards (Prestage et al., 1993), and more recently in quantum information applications using multisegmented Paul traps (Huber et al., 2008; Walther et al., 2012; Bowler et al., 2012), or an array of Penning traps (Crick et al., 2010), also in 2D configurations (Blakestad et al., 2009).

In general, a way to avoid spilling or excitation of the atoms is to perform a sufficiently slow (adiabatic) transport, but for many applications the total processing time is limited due to decoherence and an adiabatic transport may turn out to be too long. In the context of quantum information processing, transport could occupy most of the operation time of realistic algorithms, so "transport times" need to be minimized (Reichle et al., 2006; Huber et al., 2008). There are in summary important reasons to reduce the transport time, and several theoretical and experimental works have studied ways to make fast transport also faithful (Couvert et al., 2008a; Murphy et al., 2009; Masuda and Nakamura, 2010; Chen et al., 2010a; Torrontegui et al., 2011, 2012d).

### 4.1  Invariant-based Shortcuts for Transport

As done for expansions, shortcut techniques can be applied to perform fast atomic transport without final vibrational heating by combining dynamical invariants and inverse engineering. Two main scenarios can be handled in this way: shortcuts for the transport of a harmonic trap and shortcuts for the transport of an arbitrary trap. It is also possible to construct shortcuts for more complicated settings like atom stopping or launching, and combinations of transport and expansion of harmonic traps.

*Transport of a rigid harmonic trap.* – Suppose that a 1D harmonic trap should be moved from $q_0(0)$ at time $t = 0$ to $d = q_0(t_f)$ at a time $t_f$. The potential is $V = \frac{m}{2}\omega_0^2(x - q_0(t))^2$ with fixed frequency. Comparing this to Eq. (7) this implies

$$F = m\omega_0^2 q_0(t), \ \omega(t) = \omega_0, \ U = 0. \tag{54}$$

Note that Eq. (8) plays no role here and Eq. (9) becomes the only relevant auxiliary equation,

$$\ddot{q}_c + \omega_0^2(q_c - q_0) = 0, \tag{55}$$

where $q_c$ can be identified as a classical trajectory. This is the equation of a moving oscillator for which an analytical solution is known in both classical and quantum physics. From a classical mechanics point of view, the amplitude $\mathcal{A}$ of the oscillatory motion after transport is the modulus of the Fourier transform of the velocity profile associated with the trap trajectory (Couvert et al., 2008a),

$$\mathcal{A} = |\mathcal{F}[\dot{q}_0](\omega_0)| \tag{56}$$

with $\mathcal{F}[f] = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt$. This Fourier formulation of the transport problem allows for many enlightening analogies. For instance, $\mathcal{A}^2$ is mathematically identical to the intensity profile for the far field Fraunhofer diffraction pattern of an object with a transmittance having the same shape as the velocity profile for the transport. An optimal transport condition is therefore equivalent to a dark fringe in the corresponding diffraction pattern. The optimization of the conditions under which a non adiabatic transport should be carried out with a rigid harmonic trap are thus equivalent to apodization problems in optics. If the velocity profile contains the repetition of a pattern one expects an interference-like effect, this would be, for instance, the case for a symmetrical round trip transport as experimentally demonstrated in Couvert et al. (2008a).

Quantum mechanically, the wave function after transport reads

$$\Psi(q, t_f) = \tilde{\Phi}(q - q_0(t), t) \exp\left(\frac{im(q - q_0(t))\dot{q}_0}{\hbar}\right) \exp\left(\frac{i}{\hbar}\int_0^t dt' \mathcal{L}(t')\right), \tag{57}$$

where $\mathcal{L} = m\dot{q}_c^2/2 - m\omega_0^2(q_c - q_0(t))^2/2$ is the Lagrangian associated with the equation of motion (55), and $\tilde{\Phi}$ a wave function that coincides with the initial wave function at initial time and that evolves under the action of the static harmonic potential of angular frequency $\omega_0$ located at $q = q_0(0)$. Using the boundary conditions associated with the transport, one finds from Eq. (57) that an optimal transport for which the system starts in the ground state and ends up in the ground state of the displaced potential corresponds exactly to the classical criterion of a cancellation of the Fourier transform of the velocity profile, i.e. $\mathcal{A} = 0$.

Let us now address the application of invariant-based engineering. We first design an appropriate classical trajectory $q_c(t)$ fulfilling the boundary condi-

tions $q_c(0) = q_0(0) = 0$, $\dot{q}_c(0) = 0$, $\ddot{q}_c(0) = 0$ and $q_c(t_f) = q_0(t_f) = d$, $\dot{q}_c(t_f) = 0$, $\ddot{q}_c(t_f) = 0$, to ensure an evolution from the n-th state of the initial trap to the n-th state of the final trap. Then the trap motion trajectory $q_0(t)$ is deduced via Eq. (55). Some variants are vertical transport with a gravity force, so that $F = m\omega_0^2 q_0 - mg$ and Eq. (55) becomes $\ddot{q}_c + \omega_0^2(q_c - q_0) = -g$, and stopping or launching processes (Torrontegui et al., 2011).

A major concern in practice for all these applications is to keep the harmonic approximation valid. This may require an analysis of the actual potential and of the excitations taking place along the non-adiabatic transport process. Without such detailed analysis, the feasibility of the approach for a given transport objective set by the pair $d, t_f$ can be estimated by comparing lower excitation bounds (Torrontegui et al., 2012d). These are obtained using calculus of variations as we have discussed before for expansions. Writing the expectation value of potential energy for a transport mode as $\langle V(t) \rangle = \frac{\hbar\omega_0}{2}(n + 1/2) + E_P$, the time average of $E_P$ is bounded as $\overline{E_P} \geq 6md^2/(t_f^4\omega_0^2)$ (Torrontegui et al., 2011).

This scaling should be compared to the milder dependence on $t_f^{-2}$ of the time-averaged transient energy in expansions (Chen and Muga, 2010). Chen et al. (2011b) have shown how to realise this bound by allowing the discontinuous acceleration of the trap at $t = 0$ and $t = t_f$ and also finite jumps in the trap position.

In Chen et al. (2011b), the invariant-based method is complemented by optimal control theory. Since actual traps are not really harmonic, the relative displacement between the center of mass and the trap center is kept bounded as a constraint. The trajectories are then optimized according to different physical criteria: time minimization, (time-averaged) displacement minimization, and (time-averaged) transient energy minimization. The minimum time solution has a "bang-bang" form, and the minimum displacement solution is of "bang-off-bang" form. In this framework discontinuities in the acceleration $\ddot{q}_c$ at the edge times and elsewhere are allowed. Physically this means that the trap may ideally jump suddenly over a finite distance, whereas the velocity $\dot{q}_c$ and the trajectory $q_c$ remain always continuous.

*Transport of an arbitrary trap with compensating force.–* In the second main scenario, the trap potential $U(q - q_0(t))$ is arbitrary, and it is rigidly displaced along $q_0(t)$. Now, in Eq. (7), $\omega = \omega_0 = 0$, $F = m\ddot{q}_0$, and $q_c$ in Eq. (9) may be identified with the transport function $q_0$. Inverse engineering in this case is based on designing the trap trajectory $q_0$ (Torrontegui et al., 2011). In addition to $U$, there is a compensating linear potential term $-mq\ddot{q}_0$ in $H = p^2/2m - mq\ddot{q}_0 + U(q - q_0)$. The corresponding force compensates for the inertial force due to the trap motion in the rest frame of the trap, in such a way that the wave function in that frame is not modified up to a time dependent

global phase factor. This Hamiltonian was originally proposed by Masuda and Nakamura (2010) using the "fast-forward" scaling technique. Masuda (2012) has recently generalized this result for interacting, identical, spinless particles.

## 4.2 Transport of a Bose-Einstein Condensate

The two main scenarios of the previous subsection can be generalized for Bose-Einstein condensates (Torrontegui et al., 2012d). We first consider 1D harmonic transport. For the GPE

$$i\hbar\frac{\partial\psi}{\partial t}(q,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{m\omega_0^2}{2}(q - q_0(t))^2 + g_1|\psi(q,t)|^2\right]\psi(q,t), \qquad (58)$$

the results of Sec. 4.1 motivate the ansatz

$$\psi(q,t) = \exp\left\{\frac{i}{\hbar}\left(-\mu t + m\dot{q}_c q\right) - \frac{i}{\hbar}\int_0^t dt'\left[\frac{m}{2}\left(\dot{q}_c^2 - \omega_0^2(q_c^2 - q_0^2)\right)\right]\right\}\chi(\sigma), \qquad (59)$$

where $\chi(\sigma)$ satisfies the stationary GPE

$$\left[-\frac{\hbar^2}{2m}\nabla_\sigma^2 + \frac{m\omega_0^2}{2}|\sigma|^2 + U(\sigma) + g_1|\chi(\sigma)|^2\right]\chi(\sigma) = \mu\,\chi(\sigma). \qquad (60)$$

The ansatz provides indeed a solution to Eq. (58) when $q_c(t)$ satisfies Eq. (55). Inverse engineering gives the trap trajectory $q_0(t)$ from (55) after designing $q_c(t)$, as for the linear dynamics.

The inverse method can also be applied to anharmonic transport of condensates by means of a compensating force (Torrontegui et al., 2011). In either scenario this method does not require that $t_f$ satisfies any discretization condition, as it occurs with other approaches (Torrontegui et al., 2012d), and $t_f$ can in principle be made as small as desired. In practice there are of course technical and fundamental limitations (Torrontegui et al., 2011). Smaller values of $t_f$ increase the distance from the condensate to the trap center, and the effect of anharmonicity. There could be also geometrical constraints: for short $t_f$, $q_0(t)$ could exceed the interval $[0, d]$. OCT combined with the inverse method, see below, provides a way to design trajectories taking these restrictions into account.

*Optimal control theory.*— An OCT trajectory has been found when the center of the physical trap is kept inside a given range (e.g. inside the vacuum chamber), i.e. $q_\downarrow \le q_0(t) \le q_\uparrow$ (Torrontegui et al., 2012d). At the beginning

the trap is immediately set at the upper bound $q_\uparrow$ to accelerate the condensate as much as possible and at time $t_1$ the trap is moved to the lower bound $q_\downarrow$ to decelerate the condensate so as to leave it at rest at $t_f$. An important open question is to evaluate the effect of the approximate realization of the discontinuities found in the bang-bang solutions.

*Effect of Perturbations.* – Torrontegui et al. (2012d) also investigated the effect of anharmonicities when the harmonic transport protocol is applied. For a symmetrically perturbed potential $V = \omega_0^2 m \left[ (q - q_0)^2 + \alpha(q - q_0)^4 \right] /2$, the fidelity increases with increasing coupling constant $g_1$, because of the increased width of the wavefunction. They also considered that the center of the physical trap is randomly perturbed with respect to $q_0(t)$. The fidelity at $t_f$ is found to be independent of $d$ and the chosen $q_c(t)$ and increases for shorter times $t_f$ and for smaller couplings $g_1$, unlike the previous results.

## 5    Internal State engineering

Manipulating the internal state of a quantum system with time-dependent interacting fields is the basis of quantum information processing (Allen and Eberly, 1987; Vitanov et al., 2001; Bergmann et al., 1998) and many other fields. Two major routes are resonant pulses, and adiabatic methods such as "Rapid" Adiabatic Passage (RAP), Stimulated Raman Adiabatic Passage (STIRAP), and their variants. Simple fixed-area resonant pulses, such as a $\pi$ pulse, may be fast if intense enough, but they are also highly sensitive to variations in the pulse area, and to inhomogeneities in the sample (Allen and Eberly, 1987). Composite pulses provide an alternative to the single $\pi$ pulse, with some successful applications (Lewvitt, 1986; Collin et al., 2004; Torosov et al., 2011), but still they need an accurate control of pulse phase and intensity. In NMR, composite pulses are being superseded by adiabatic passage methods, which have also been very successful in laser cooling, chemical reaction dynamics, metrology, atom optics, interferometry, or cavity quantum electrodynamics. Adiabatic passage is robust versus parameter variations but slow. It is moreover prone to decoherence because of the effect of noise over the long times required. This motivates the search for fast and robust shortcuts, with respect to parameter variations and noise.

Several methods to find STA have been put forward for two- and three-level atomic systems. Among them, methods that we have already discussed in Sec. 2, like the transitionless driving, invariant-based engineering, or OCT.

Using the convention $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, assume a two-level system with a Hamiltonian of the form

$$H_0(t) = \frac{\hbar}{2} \begin{pmatrix} -\Delta(t) & \Omega_R(t) - i\Omega_I(t) \\ \Omega_R(t) + i\Omega_I(t) & \Delta(t) \end{pmatrix}. \tag{61}$$

In quantum optics it describes the semiclassical coupling of two atomic levels with a laser in a laser-adapted interaction picture, where $\Omega_c(t) = \Omega_R(t) + i\Omega_I(t)$ is the complex Rabi frequency and $\Delta(t)$ the time-dependent detuning between laser and transition frequencies. We will keep the language of the atom-laser interaction hereafter, but in other two-level systems, for example, in a spin-1/2 system or in a Bose-Einstein condensate on an accelerated optical lattice (Bason et al., 2012), $\Omega_c(t)$ and $\Delta(t)$ may correspond to different physical quantities.

Initially at time $t = 0$, the atom is assumed to be in the ground state $|1\rangle$. The goal is to achieve a perfect population inversion such that at a time $t = T$ the atom is in the excited state. For a $\pi$ pulse the laser is on resonance, i.e. $\Delta(t) = 0$ for all $t$. If the Rabi frequency is chosen like $\Omega_c(t) = |\Omega_c(t)| e^{i\alpha}$, with a time-independent $\alpha$, and such that $\int_0^T dt \, |\Omega_c(t)| = \pi$, the population is inverted at time $T$. A simple example is the "flat" $\pi$ pulse with $\Omega_c(t) = e^{i\alpha}\pi/T$.

Adiabatic schemes provide another major route for population inversion. In the "Rapid Adiabatic Passage" technique the radiation is swept slowly through resonance. The term "rapid" here means that the frequency sweep is shorter than the life-time of spontaneous emission and other relaxation times. Many schemes corresponding to different functions $\Delta(t)$, and $\Omega_R(t)$ are possible. The simplest is a Landau-Zener approach, with $\Delta$ linear in time, $\Omega_R$ constant and $\Omega_I = 0$.

*Transitionless shortcuts to adiabaticity.–* If an adiabatic scheme is used and the adiabaticity condition $\frac{1}{2}|\Omega_a| \ll |\Omega(t)|$ (where $\Omega = \sqrt{\Delta^2 + \Omega_R^2}$, $\Omega_I = 0$, and $\Omega_a \equiv [\Omega_R\dot{\Delta} - \dot{\Omega}_R\Delta]/\Omega^2$) is not fulfilled, the inversion fails. We may still get an inversion (i.e. a shortcut) by applying a counterdiabatic field such that its maximum is not larger than the maximum of $\Omega_R$ (Chen et al., 2010c). The total Hamiltonian, see Sec. 2.2, for the transitionless shortcut protocol is (Demirplak and Rice, 2008; Berry, 2009; Chen et al., 2010c)

$$H_{0a}(t) = \frac{\hbar}{2} \begin{pmatrix} -\Delta & \Omega_R - i\Omega_a \\ \Omega_R + i\Omega_a & \Delta \end{pmatrix}. \tag{62}$$

*Invariant-based Shortcuts.* – STA in two-level systems can be also found making use of Lewis-Riesenfeld invariants (Fasihi et al., 2012; Ruschhaupt et al., 2012). For $H_0$ in Eq. (61), a dynamical invariant may be parameterized as

$$I(t) = \frac{\hbar}{2}\mu \begin{pmatrix} \cos(\Theta(t)) & \sin(\Theta(t))\, e^{-i\alpha(t)} \\ \sin(\Theta(t))\, e^{i\alpha(t)} & -\cos(\Theta(t)) \end{pmatrix}, \tag{63}$$

where $\mu$ is a constant with units of frequency to keep $I(t)$ with dimensions of energy. From the invariance condition the functions $\Theta(t)$ and $\alpha(t)$ must satisfy

$$\dot{\Theta} = \Omega_I \cos\alpha - \Omega_R \sin\alpha, \tag{64}$$
$$\dot{\alpha} = -\Delta(t) - \cot\Theta\,(\Omega_R \cos\alpha + \Omega_I \sin\alpha).$$

The eigenvectors of the invariant are

$$|\phi_+(t)\rangle = \begin{pmatrix} \cos(\Theta/2)\, e^{-i\alpha/2} \\ \sin(\Theta/2)\, e^{i\alpha/2} \end{pmatrix}, \quad |\phi_-(t)\rangle = \begin{pmatrix} \sin(\Theta/2)\, e^{-i\alpha/2} \\ -\cos(\Theta/2)\, e^{i\alpha/2} \end{pmatrix}, \tag{65}$$

with eigenvalues $\pm\frac{\hbar}{2}\mu$. A general solution $|\Psi(t)\rangle$ of the Schrödinger equation can be written as a linear combination

$$|\Psi(t)\rangle = c_+ e^{i\kappa_+(t)} |\phi_+(t)\rangle + c_- e^{i\kappa_-(t)} |\phi_-(t)\rangle, \tag{66}$$

where $c_\pm$ are complex, constant coefficients, and $\kappa_\pm$ are the phases of Lewis and Riesenfeld (1969) introduced in Eq. (4). Let $\gamma = -2\kappa_+ = 2\kappa_-$, then $\gamma$ must be a solution of

$$\dot{\gamma} = \frac{1}{\sin\Theta}\,(\cos\alpha\,\Omega_R + \sin\alpha\,\Omega_I). \tag{67}$$

Equivalently a solution of the Schrödinger equation $|\Psi(t)\rangle$ may be designed with the same parameterization as above ($|\Psi(t)\rangle\langle\Psi(t)|$ is a dynamical invariant.) and, by putting this ansatz into the Schrödinger equation, Eqs. (64) and (67) are found.

If $\Omega_R(t)$, $\Omega_I(t)$ and $\Delta(t)$ are given, Eqs. (64) and (67) have to be solved to get $\Theta(t)$, $\alpha(t)$ and $\gamma(t)$. A particular solution of the Schrödinger equation is then given by

$$|\psi(t)\rangle = |\phi_+(t)\rangle e^{-i\gamma(t)/2}. \tag{68}$$

To find invariant-based shortcuts and inverse engineer the Hamiltonian $\Theta(t)$, $\alpha(t)$, and $\gamma(t)$ are fixed first, fulfilling the boundary conditions $\Theta(0) = 0$ and $\Theta(T) = \pi$. The wave function (68) corresponds then to an atom in the ground state at $t = 0$ and in the excited state at $t = T$, i.e. a perfect population inversion. Then, by inverting (64) and (67),

$$\Omega_R = \cos\alpha \sin\Theta\, \dot{\gamma} - \sin\alpha\, \dot{\Theta}, \tag{69}$$
$$\Omega_I = \sin\alpha \sin\Theta\, \dot{\gamma} + \cos\alpha\, \dot{\Theta}, \tag{70}$$
$$\Delta = -\cos\Theta\, \dot{\gamma} - \dot{\alpha}. \tag{71}$$

There is much freedom in designing such a shortcut because the auxiliary functions $\Theta(t)$, $\alpha(t)$ and $\gamma(t)$ can be chosen arbitrarily except for the boundary conditions.


## 5.2   Effect of Noise and Perturbations


A key aspect to choose among the many possible shortcuts is their stability or robustness versus different perturbations. Ruschhaupt et al. (2012) have derived optimal invariant-based shortcut protocols, maximally stable concerning amplitude noise of the interaction and with respect to systematic errors. It turns out that the perturbations due to noise and systematic errors require different optimal protocols.

Let the ideal, unperturbed Hamiltonian be the $H_0(t)$ of Eq. (61). In Ruschhaupt et al. (2012), it is assumed that the errors affect $\Omega_R$ and $\Omega_I$ but not the detuning $\Delta$, which, for an atom-laser realization of the two-level system is more easily controlled.

For systematic errors, for example if different atoms at different positions are subjected to slightly different fields due to the Gaussian shape of the laser, the actual, experimentally implemented Hamiltonian is $H_{01} = H_0 + \beta H_1$, where $H_1(t) = H_0(t)\big|_{\Delta \equiv 0}$ and $\beta$ is the amplitude of the systematic error.

The second type of error considered in Ruschhaupt et al. (2012) is amplitude noise, which is assumed to affect $\Omega_R$ and $\Omega_I$ independently with the same strength parameter $\lambda^2$. This is motivated by the assumption that two lasers

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may be used to implement the two parts of the Rabi frequency. The final master equation describing systematic error and amplitude-noise error is

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[H_0 + \beta H_1, \hat{\rho}] - \frac{\lambda^2}{2\hbar^2}\left([H_{2R}, [H_{2R}, \hat{\rho}]] + [H_{2I}, [H_{2I}, \hat{\rho}]]\right), \tag{72}$$

where $H_{2R}(t) = H_0(t)\big|_{\Delta \equiv \Omega_I \equiv 0}$, and $H_{2I}(t) = H_0(t)\big|_{\Delta \equiv \Omega_R \equiv 0}$.

Before studying both types of error together it is fruitful to look at them separately.

*Amplitude-Noise Error.*– If there is no systematic error ($\beta = 0$) and only an amplitude-noise error affecting the Rabi frequencies, a noise sensitivity can be defined as

$$q_N := -\frac{1}{2}\frac{\partial^2 P_2}{\partial \lambda^2}\bigg|_{\lambda=0} = -\frac{\partial P_2}{\partial(\lambda^2)}\bigg|_{\lambda=0},$$

where $P_2$ is the probability to be in the excited state at final time $T$, i.e. $P_2 \approx 1 - q_N \lambda^2$.

To find an invariant-based shortcut protocol maximally stable concerning amplitude noise, it is first assumed that the unperturbed solution (68) satisfies $\Theta(0) = 0$ and $\Theta(T) = \pi$. Using a perturbation approximation of the solution and keeping only terms up to $\lambda^2$ (Ruschhaupt et al., 2012),

$$\begin{aligned} q_N = \frac{1}{4}\int_0^T dt \Big[ & (\cos^2\Theta + \cos^2\alpha\sin^2\Theta)(m\sin\alpha - \cos\alpha\dot{\Theta})^2 \\ & + (\cos^2\Theta + \sin^2\alpha\sin^2\Theta)(m\cos\alpha + \sin\alpha\dot{\Theta})^2 \Big], \end{aligned} \tag{73}$$

where $m(t) = -\dot{\gamma}\sin\Theta$. Minimizing the error sensitivity $q_N$ by Euler-Lagrange one gets that the optimal solutions satisfy (Ruschhaupt et al., 2012) $\alpha = n\pi/4$, $n$ odd, and

$$(3 + \cos(2\Theta))\ddot{\Theta} = \sin(2\Theta)(\dot{\Theta})^2. \tag{74}$$

The corresponding $\Omega_R$ and $\Omega_I$ can be calculated from Eqs. (69) and (70). In this case, $\Omega_R = \pm\dot{\Theta}/\sqrt{2} = \pm\Omega_I$ and $\Delta(t) = 0$. The optimal noise sensitivity value is $q_N = 1.82424/T < \pi^2/(4T)$ and the maximum of the Rabi frequency is $\Omega_R(t_f/2)t_f \approx 2.70129$. An approximate solution of Eq. (74) is given by $\Theta(t) = \pi t/T - \frac{1}{12}\sin(2\pi t/T)$, with a noise sensitivity of $q_N = 1.82538/T$.

*Systematic Error.*— If there is no amplitude-noise error ($\lambda = 0$) and only systematic error, a systematic error sensitivity is defined as

$$q_S := -\frac{1}{2}\frac{\partial^2 P_2}{\partial\beta^2}\bigg|_{\beta=0} = -\frac{\partial P_2}{\partial(\beta^2)}\bigg|_{\beta=0},$$

where $P_2$ is as before the probability to find the atom in the excited state at final time $T$. $q_S$ may be calculated with a perturbation approximation of the solution keeping only terms up to $\beta^2$ (Ruschhaupt et al., 2012).

To find an optimal scheme the invariant based technique is used again. The evolution of the unperturbed state can be parameterized as before, $|\psi(t)\rangle$ (see Eq. (68)), with the boundary values $\Theta(0) = 0$ and $\Theta(T) = \pi$. The expression for the systematic error sensitivity is now

$$q_S = \left|\int_0^T dt\, e^{-i\gamma}\dot\Theta\sin^2\Theta\right|^2.$$

The optimal value is clearly $q_S = 0$. An example of a class which fulfills $q_S = 0$ is found by letting $\gamma(t) = n\left(2\Theta - \sin(2\Theta)\right)$. If follows that $q_S = \sin^2(n\pi)/(4n^2)$, so for $n = 1, 2, 3, ..., q_S = 0$. There is still some freedom left, this allows further optimization concerning additional constraints.

*Systematic and amplitude-noise errors.–* If both errors coexist the optimal schemes will depend on their relative importance. Ruschhaupt et al. (2012) examine numerically the behavior of different protocols. Fig. 1 shows that the different optimal schemes perform better than the other one depending on the dominance of one or the other type of error.

Additional work is in required to extend the results in Ruschhaupt et al. (2012) to different types of noise and perturbations. Apart from the invariant-based approach, Lacour et al. (2008) have proposed robust trajectories in the adiabatic parameter space that maximize the population transfer for a two-level system subjected to dephasing. Also the robustness of the "parallel adiabatic passage" technique (keeping the eigenvalues of the Hamiltonian parallel (Dridi et al., 2009)) with respect to fluctuations of the phase, amplitude and pulse area was analyzed in Gulérin et al. (2011).

## 5.3 Three-level Systems

The transitionless driving for stimulated rapid adiabatic passage from level 1 to level 3 in a lambda configuration with an intermediate state 2 making use of a pumping and a Stokes laser was studied in Unayan et al. (1997); Demirplak and Rice (2003, 2005); Chen et al. (2010c). The fast-driving cd field connects levels $|1\rangle$ and $|3\rangle$. This implies in general a weak magnetic dipole transition, which limits the ability of the field to shorten the times. Invariant-
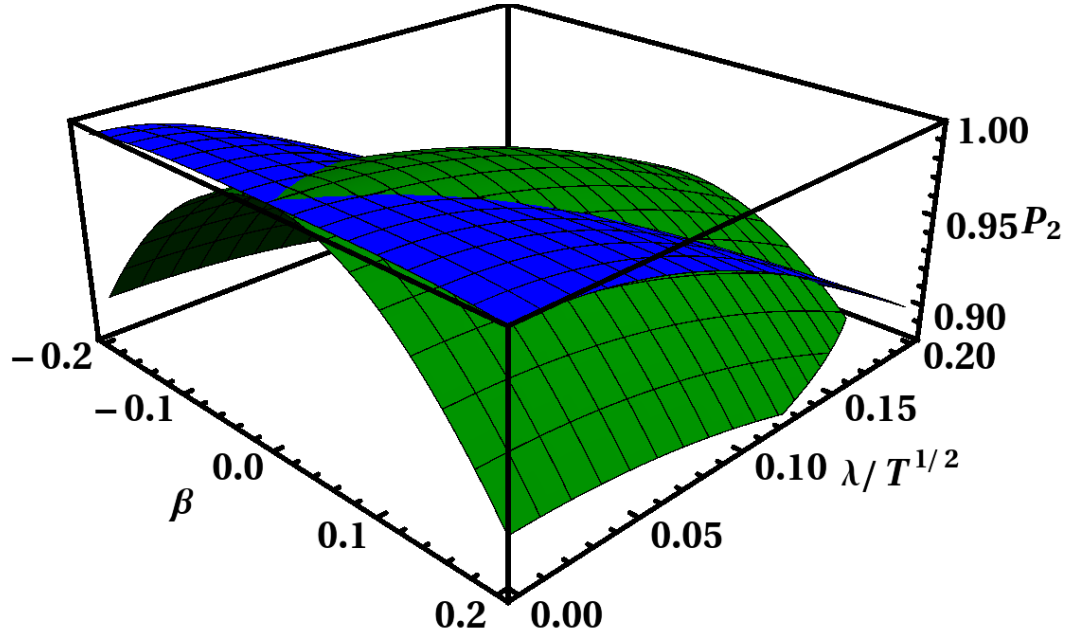
Fig. 1. **(Color online) Probability $P_2$ versus noise error and systematic error parameter; optimal systematic stability protocol (blue), optimal noise protocol (green).**

based engineering solves the problem by providing alternative shortcuts that do not couple directly levels $|1\rangle$ and $|3\rangle$ (Chen and Muga, 2012), as discussed below. It should be noted though that in an optical analogy of STA to engineer multimode waveguides all these schemes (with or without 1-3 coupling) may in principle be implemented (Lin et al., 2012; Tseng and Chen, 2012) by computer-generated holograms. In this analogy, based on the paraxial approximation, space plays the role of time so that the effect of the shortcuts is to shorten the length of the mode converters.

In Chen and Muga (2012), using two lasers on resonance with the 1-2 and 2-3 transitions, two single-mode protocols that make use of one eigenstate of the invariant are described. In these protocols full fidelity requires an infinite laser intensity, and shortening the time also implies an energy cost. The first protocol, based on simple sine and cosine functions for the pumping and Stokes lasers keeps the population of level 2 small. To achieve the same fidelity, less intensity is required in the second protocol, in which the intermediate level $|2\rangle$ is populated. The population of the intermediate level is usually problematic when its time decay scale is smaller than the process time. While this may be a serious drawback for an adiabatic slow process, it need not be for a fast shortcut. Protocols that populate level 2 may thus be considered as useful alternatives for certain systems and sufficiently short process times.

In the previous two protocols the initial state is not exactly $|1\rangle$ to avoid a divergence in the Rabi frequency. A third multi-mode wave-function protocol

is also proposed in Chen and Muga (2012) using the same fields as for the first protocol but with an initial state which is simply the bare state $|1\rangle$. It provides a much less costly shortcut so exploring the multi-mode approach for this and other systems is an interesting task for future work.

Inverse engineering of four-level systems has been considered in Güngördü et al. (2012), where a full Lie-algebraic classifffication and detailed construction of the dynamical invariants is provided.

## 5.4  Spintronics

Coherent spin manipulation in quantum dots is the key element in the state-of-the-art technology of spintronics. Ban et al. (2012) have considered the electric control of electron spin in a quantum dot formed in a two-dimensional electron gas confined by the material composition under a weak magnetic field, focusing on the spin flip in the doublet of the lowest orbital state. The influence from higher orbital states can be taken into account by the Löwdin partition technique reducing the full Hamiltonian into an effective two-level one in which the matrix elements depend on electric field components. Using invariant-based inverse engineering the time-dependent electric fields are designed so to flip the spin rapidly and avoid decoherence effects. The results are stable with respect to environmental noise and the device-dependent noise and may open new possibilities for high-fidelity spin-based quantum information processing.

## 5.5  Experiments

The counterdiabatic or transitionless approach described in Secs. 2.2, 2.4, and 5.1 has been applied recently to invert the population of different two-level systems:

In Bason et al. (2012) the effective two-level system is set as a condensate in the bands of an accelerated optical lattice Zenesini et al. (2009). Writting the Hamiltonian in Cartesian-like coordinates as $H = X\sigma_x + Y\sigma_y + Z\sigma_z$, $X$ may be controlled by the trap depth, $Z$ by the lattice acceleration Zenesini et al. (2009), and $Y$ could in principle be implemented by a second shifted lattice. The counterdiabatic term in Eqs. (28) or (62) is of the form $Y\sigma_y$ whose realization is cumbersome in this setting. The alternative was to perform a unitary transformation that leads to the same final state modifying the original $X$ and $Z$ terms. This manipulation, discussed in Sec. 2.4. (see Eq. (31), was interpreted as a $Z$-rotation in Ibáñez et al. (2012a), where it is compared

to the one based on the first-order superadiabatic cd-term $H_{cd}^{(1)}$.[8] Landau-Zener and a "tangent" protocol with a tangent function for $Z$ unnafected by the rotation, are used as a reference, the later being found to be very robust versus a simulated variation of control parameters.

In Zhang et al. (2012b) the two-level system is a single nitrogen vacancy center in diamond controlled by time dependent microwave fields. The reference process is a Landau-Zener transition, and the $Y\sigma_y$ cd-term is implemented by a field oscillating $\pi/2$ radians out of phase with respect to the field that provides the $X\sigma_x$ term. As the maximal value of the total field amplitude is bounded, in this case to avoid undesired transitions, a "rapid-scan" approach is implemented to shorten the protocol time: the protocol is divided into a discrete set of time segments with varying phase and the time duration of each segment is adjusted so that the maximal amplitude allowed is applied.

## 6 Wavepacket Splitting

Splitting a wavefunction without exciting it is important in matter wave interferometry (Hohenester et al., 2007; Grond et al., 2009a,b; Pezze et al., 2005). For linear waves, described by the Schrödinger equation, it is a peculiar operation, as adiabatic following is not robust but unstable with respect to a small external potential asymmetry (Gea-Banaloche, 2002; Torrontegui et al., 2012b). The ground-state wavefunction "collapses" into the slightly lower well so that a very slow trap potential bifurcation fails to split the wave except for perfectly symmetrical potentials. A fast bifurcation with a rapidly growing separating potential succeeds to split the wave but at the price of a strong excitation. STA that speed up the adiabatic process along a non-adiabatic route overcome these problems (Torrontegui et al., 2012b). Numerical modelling shows that the wave splitting via shortcuts is significantly more stable than the adiabatic following with respect to asymmetric perturbations and avoids the final excitation. Specifically Torrontegui et al. (2012b) use the streamlined version (Torrontegui et al., 2012a) of the fast-forward technique of Masuda and Nakamura (2010), see Sec. 2.3, applied to Gross-Pitaevskii or Schrödinger equations after having found some obstacles to apply the invariants-based method (the eigenvectors of quadratic-in momentum invariants do not satisfy the required boundary conditions (Torrontegui et al., 2012a)), and the transitionless-driving algorithm (Demirplak and Rice, 2003) (because of difficulties to implement in practice the counter-diabatic terms). The following discussion refers to the Schrödinger equation except for a final comment on the GPE.

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[8] The use of the term "superadiabatic" in Bason et al. (2012) differs -is broader there- from the one in Sec. 2.4.

*Fast-forward approach.* — To apply the FF approach the density $r(x, t)$ must be first designed. Assume the splitting of an initial single Gaussian $f(x, 0) = e^{-x^2/(2a_0^2)}$, where $a_0$ is the width of the ground state for a harmonic oscillator with frequency $\omega/\hbar$, $a_0 = \sqrt{\hbar/(m\omega)}$, into a final double Gaussian $f(x, t_f) = e^{-(x-x_f)^2/(2a_0^2)} + e^{-(x+x_f)^2/(2a_0^2)}$. The interpolation

$$r(x, t) = z(t)[e^{-(x-x_f(t))^2/(2a_0^2)} + e^{-(x+x_f(t))^2/(2a_0^2)}], \tag{75}$$

where $z(t)$ is a normalization function, generates simple $Y$-shaped potentials. The conditions $\dot{x}_0(0) = \dot{x}_0(t_f) = 0$ are imposed, so $\dot{r} = 0$ at boundary times. In Torrontegui et al. (2012b) $x_0(s) = x_f(3s^2 - 2s^3)$, where $s = t/t_f$, is chosen for simplicity, and Eq. (25) is solved for the initial conditions to get the FF potential with Eq. (23).

*Effect of the perturbation.* — The effects of an asymmetric perturbation may be studied with the potential $V_\lambda = V_{FF} + \lambda\theta(x)$, where $\theta$ is the step function and $V_{FF}$ the potential obtained via Eqs. (23), (25), and (75) with $\lambda = 0$. The goal is to find a stable time-dependent porotocol that, even without knowing the value of $\lambda$, is able to produce the split state.

*Moving two-mode model.* — Static two-mode models have been used before to analyze splitting processes (Javanainen and Ivanov, 1999; Grond et al., 2009b; Aichmayr, 2010). Torrontegui et al. (2012b) consider instead a two-level model with moving left and right basis functions to provide analytical estimates and insight as a complement of the more detailed FF approach.

Assume first the (symmetrical and orthogonal) moving left and right bare basis states $|L(t)\rangle = \binom{0}{1}$, $|R(t)\rangle = \binom{1}{0}$, and a corresponding two-mode Hamiltonian model

$$H(t) = \frac{1}{2}\begin{pmatrix} \lambda & -\delta(t) \\ -\delta(t) & -\lambda \end{pmatrix}, \tag{76}$$

where $\delta(t)/\hbar$ is the tunneling rate (Javanainen and Ivanov, 1999; Grond et al., 2009b) and $\lambda$ the energy difference between the two wells (Aichmayr, 2010). We may simply consider $\lambda$ constant through a given splitting process and equal to the perturbative parameter that defines the asymmetry. Thus, the instantaneous eigenvalues are $E_\lambda^\pm(t) = \pm\frac{1}{2}\sqrt{\lambda^2 + \delta^2(t)}$, and the normalized eigenstates

$$|\psi_\lambda^+(t)\rangle = \sin\left(\tfrac{\alpha}{2}\right)|L(t)\rangle - \cos\left(\tfrac{\alpha}{2}\right)|R(t)\rangle,$$
$$|\psi_\lambda^-(t)\rangle = \cos\left(\tfrac{\alpha}{2}\right)|L(t)\rangle + \sin\left(\tfrac{\alpha}{2}\right)|R(t)\rangle, \tag{77}$$

where $\tan\alpha = \delta(t)/\lambda$ defines the mixing angle.

When $\{|L(t)\rangle, |R(t)\rangle\}$ are close enough initially (and $\delta(0) \gg \lambda$), the instantaneous eigenstates of $H$ are close to the symmetric ground state $|\psi_0^-(0)\rangle =$

$\frac{1}{\sqrt{2}}(|L(0)\rangle + |R(0)\rangle)$ and the antisymmetric excited state $|\psi_0^+(0)\rangle = \frac{1}{\sqrt{2}}(|L(0)\rangle - |R(0)\rangle)$ of the single well. At $t_f$ two extreme regimes may be distinguished: *i)* For $\delta(t_f) \gg \lambda$ the final eigenstates of $H$ tend to $|\psi_\lambda^\mp(t_f)\rangle = \frac{1}{\sqrt{2}}(|L(t_f)\rangle \pm |R(t_f)\rangle)$ which correspond to the symmetric and antisymmetric splitting states. *ii)* For $\delta(t_f) \ll \lambda$ the final eigenfunctions of $H$ collapse and become right and left localized states: $|\psi_\lambda^-(t_f)\rangle = |L(t_f)\rangle$ and $|\psi_\lambda^+(t_f)\rangle = |R(t_f)\rangle$. Since $\delta(t_f)$ is set as a small number to avoid tunnelling in the final configuration, the transition from one to the other regime explains the collapse of the ground state function to one of the wells at small $\lambda \approx \delta(t_f)$.

*Dynamics of the two-mode model.—* In a moving-frame interaction-picture wave function $\psi^A = A^\dagger \psi^S$, where $A = \sum_{\beta=L,R} |\beta(t)\rangle\langle\beta(0)|$ and $\psi^S$ is the Schrödinger-picture wave function, $\psi^A$ obeys $i\hbar\dot{\psi}^A = (H_A - K_A)\psi^A$, with $H_A = A^\dagger H A$, and $K_A = i\hbar A^\dagger \dot{A}$. For real $\langle x|R(t)\rangle$ and $\langle x|L(t)\rangle$, the symmetry $\langle x|R(t)\rangle = \langle -x|L(t)\rangle$ makes $K_A = 0$.

We may invert Eq. (77) to write the bare states in terms of ground and excited states, and get $\delta(t)$ from Eq. (76). The actual dynamics is approximated by identifying $|\psi_0^\pm(t)\rangle$ and $E_0^\pm(t)$ with the instantaneous ground and excited states and energies of the unperturbed FF Hamiltonian. They are combined to compute the bare basis in coordinate representation, and with them the matrix elements $\langle\beta'|H_\lambda|\beta\rangle$. The dynamics in the moving frame for the two-mode Hamiltonian may then be solved.

*Sudden approximation.—* The behaviour at low $\lambda$ may be understood with the sudden approximation (Messiah, 1999). Its validity requires (Messiah, 1999) $t_f \ll \hbar/\Delta\overline{H_A}$, where $\Delta\overline{H_A} = \sqrt{\langle\psi(0)|\overline{H_A}^2|\psi(0)\rangle - \langle\psi(0)|\overline{H_A}|\psi(0)\rangle^2}$ and $\overline{H_A} = \frac{1}{t_f}\int_0^{t_f} dt' H_A(t')$. With $|\psi(0)\rangle = |\psi_0^-(0)\rangle$ the condition to apply the sudden approximation becomes $\lambda \ll \frac{2\hbar}{t_f}$. In this regime the dynamical wave function $\psi(t_f)$ is not affected by the perturbation and becomes the ideal split state $\psi_0^-(t_f)$, up to a phase factor.

The previous results may be extended to weakly non-linear Bose-Einstein condensates for which $g_1/a_0\lambda \ll 1$. Otherwise the instability of adiabatic splitting with respect to perturbations is strongly suppressed by the compensating effect of the non-linear term (Torrontegui et al., 2012b). Of course the shortcuts would still be useful if the time is to be reduced.

# 7   Discussion

We have presented an overview of recent work on shorcuts to adiabaticity (STA) covering a broad span of methods and physical systems. STA offer

many promising research and application avenues with practical and fundamental implications. Several pending tasks have been described along the text. We add here some more: To extend the set of basic physical limitations and laws for fast processes in specific operations, taking into account different constraints; To generate simple, viable shortcuts making systematic use of symmetries; To enhance robustness versus different types of noise and perturbations; To perform inverse-engineering with invariants beyond the quadratic-in-momentum family; To develop shortcuts for adiabatic computing, and in general for Hamiltonians that cannot be easily diagonalized, as in Nehrkorn et al. (2011); To design or supplement STA by optimal control theory methods. We have seen some exaples but many other optimization problems await unexplored.

Indeed STA open interesting prospects to improve or make realizable quantum information and technology operations, by implementing new fast and robust transport or expansion approaches, internal state manipulations, and cooling protocols; nuclear magnetic resonance is another field where developing ideal pulses may benefit from STA. STA could be also useful beyond single or many-body quantum systems, e.g. to build optical short-length mode converters or for designing mechanical operations with nanoparticles, mesoscopic, or macroscopic objects. In classical mechanics there are many examples of adiabatic evolution that may be shortcut. The application of this concept to interacting classical gas manipulation remains also an open question.

We have witnessed in a few years a surge of activity and applications that could have hardly been predicted. Researchers creativity will likely continue to surprise us in the stimulating crossroad of STA with new, unnexpected concepts and applications.

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