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A more accurate theory of a flexible-beam pendulum

Gilles Dolfo[a)] and Jacques Vigué
Laboratoire Collisions Agrégats Réactivité-IRSAMC, Université de Toulouse-UPS and CNRS UMR 5589
118, Route de Narbonne, 31062 Toulouse Cedex, France

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A pendulum suspended by a flexible beam made of a thin metal strip is commonly used in clocks. However, the usual theory describing its motion is approximate and incomplete. We first recall that simple theory and we then present a more complete theory, which describes the shape of the flexible beam by elasticity theory. We find that the pendulum has two resonant frequencies, corresponding to different shapes of the flexible beam and different motions of the pendulum. The dynamical effects of the flexible beam are directly related to its dimensions and its Young's modulus.
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I. INTRODUCTION

Since the work of Galileo Galilei,[1] the pendulum has played an important role in physics.[2,3] There are many reasons for this importance: the pendulum is a quasi-harmonic oscillator; it is easy to build a pendulum with a large quality factor Q (i.e., with a very sharp resonance); and the theoretical analysis of pendulum motion can involve many degrees of refinement.[4] Hundreds of papers with the word "pendulum" in their titles have been published in journals devoted to the teaching of physics.

However, amongst this vast literature, we have not found a complete theory of the pendulum suspended by a flexible beam. In this paper, we therefore present a simplified theory of such a pendulum in Sec. II. Then, in Sec. III, we develop a more complete theory in which the shape of the flexible beam is described by elasticity theory. After our analysis was completed, we found that a similar study had been published by Le Rolland[5] in 1923. As access to this old paper is not easy, we have nevertheless decided to present this theory.

The interest in this type of pendulum is due to the fact that such a suspension is commonly used in pendulum clocks (see Fig. 1); the pendulum is made of a long rod with a heavy bob attached near its lower end. The flexible "beam" consists of a thin metal strip, which is clamped at both ends, the lower end in the top of the rod and the upper end in a stable frame. The flexible beam is called a "spring" by clock makers and we will use this term throughout this paper. It is usually difficult to see the spring of a clock because it is hidden by the mechanism, but the interested reader will find many images of clock suspension springs on the Web. The springs are made of an elastic metal like spring steel or Elinvar/NispanC, a metal with a Young's modulus almost independent of temperature.[6] For more details, we refer the reader to the book of Matthys.[7] In our calculations, we will assume that this spring can only flex and that it cannot stretch.

The theory described in Sec. III predicts two resonances, while the simplified theory predicts only one. There is a low-frequency resonance associated with the usual pendular motion and a high-frequency resonance with a rolling motion.

Finally, in Sec. IV, we compare the present results to related works by Hughes[8] and by Gleiser.[9] Hughes predicted (theoretically) and verified (experimentally) the double resonance characteristics for a pendulum made of a sphere suspended by an infinitely soft string. He explained this result as a particular case of a double pendulum. Gleiser used a variational formalism to describe a pendulum with arbitrary elastic properties and mass distribution. His results differ from ours due to a failure in his variational calculation.

II. SIMPLE THEORY OF A PENDULUM WITH A SPRING

Figure 1 shows a schematic drawing of the pendulum and defines our notation. The spring mass is usually considerably smaller than the pendulum mass so it is a good approximation to neglect the inertia of the spring. We will do so in all our calculations.

The pendulum oscillates in the xy-plane and its position is measured by the angle θ between the vertical and the line

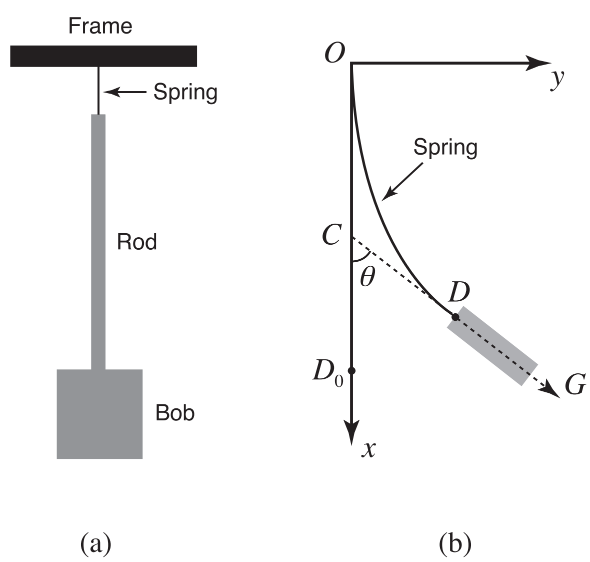

Fig. 1. (a) Schematic drawing of the pendulum with the spring clamped in the frame and at the top of the rod, which carries the pendulum bob. (b) Enlarged drawing of the spring oscillating in the xy-plane. The spring is clamped at O and at D, with D_0 being the position of D when the pendulum is at equilibrium. In the case of small oscillations, the shape of the spring (the curve OD) is assumed to be a part of a circle and C is at the midpoint of OD_0. The angle θ, which is assumed to be small in our calculations, has been magnified for clarity. The center-of-mass of the pendulum body is at G (far from D on the scale of the diagram).

segment DG. If the spring length l is small enough, it is reasonable to assume that the spring radius of curvature R is uniform. Then curve OD is a circular arc and the angle θ satisfies $l = R\theta$. The line DG, which is tangent to the circle at D, intersects the x-axis at C. Using elementary geometry, we find the coordinate x_C of C:

$$x_C = l\frac{1-\cos\theta}{\theta\sin\theta} \approx \frac{l}{2}\left[1+\frac{\theta^2}{12}\right]. \tag{1}$$

At first order in θ, the point C is fixed at the midpoint of the spring when the spring is at equilibrium. For small oscillations, the pendulum motion is a rotation around point C. The elastic torque is equal to $-K\theta$, to first order in θ, where K is the spring stiffness. The torque exerted by gravity is $-Mg(h+l/2)\sin\theta \approx -Mg(h+l/2)\theta$, where M is the pendulum mass, g is the local gravitational field strength, and h is the length of DG, so that $CG = h + (l/2)$ (see Fig. 1). The equation of motion is

$$I_C\frac{d^2\theta}{dt^2} \approx -\left[Mg\left(h+\frac{l}{2}\right)+K\right]\theta, \tag{2}$$

where I_C is the pendulum moment of inertia calculated for rotation around point C. By Huygens's theorem (i.e., the parallel-axis theorem), we have $I_C = M[(h+l/2)^2+\rho^2]$, where ρ is the gyration radius of the pendulum body. The motion is therefore harmonic with the angular frequency

$$\Omega = \sqrt{\frac{g(h+l/2)+K/M}{(h+l/2)^2+\rho^2}}. \tag{3}$$

If the elastic torque is negligible with respect to the gravitational torque, i.e., if $K/M \ll g(h+l/2)$, then the angular frequency Ω given by Eq. (3) is the usual formula for a gravity pendulum. When the elastic torque is not negligible, Ω increases with the stiffness K.

However, this theory is not satisfactory for two reasons:

- the range of validity of the assumption describing the spring shape as a circular arc is unknown;
- the spring stiffness K has been introduced without any information on its value. In order to fully describe the pendulum, it is necessary to relate K to the spring dimensions and to the Young's modulus of its material.

III. A MORE COMPLETE THEORY OF A PENDULUM WITH A SPRING

In elasticity theory, the spring is described as a flexible beam, with its shape given by the Euler-Bernoulli equation. The equation derived in many textbooks[10] uses an equilibrium theory, which neglects spring inertia. This approximation should be very good because the spring is considerably lighter than the pendulum; the theory's validity can be judged afterwards by verifying that the pendulum resonance frequencies are considerably smaller than those of the spring. The calculation of these resonance frequencies is similar to the calculation of those of piano strings (see, for instance, Ref. 11 and references therein). Throughout the present calculation, we will use a first-order approximation in the oscillation amplitude. Our calculation involves two steps:

- first, we calculate the shape of the spring, assuming that the force and the torque exerted by the pendulum on the spring are known;
- second, we use these results in the equations of motion of the pendulum and we obtain two coupled linear differential equations.

A. First step: Calculation of the spring shape via the Euler-Bernoulli equation

As shown in Fig. 2, the spring bends in the xy-plane. The Euler-Bernoulli equation relates the radius of curvature R of the spring at any point N to the z-component of the torque $\tau_z(N)$ exerted on the spring at this point:

$$\tau_z(N) = \frac{\mu}{R}, \tag{4}$$

with

$$\mu \equiv EI_s. \tag{5}$$

Here E is the Young's modulus of the spring material and I_s is the second moment of the area of its cross section. When the spring is at its equilibrium position along the x-axis, I_s is given by $I_s = \int\int y^2 dydz$, with the integration extending over the spring cross section centered at $y = 0$, the z-direction being perpendicular to the xy-plane. For a rectangular section of width c in the xy-plane and length d along z, $I_s = c^3 d/12$. Only the z-component of the torque $\tau_z(N)$ is nonzero, because of our assumption of motion in the xy-plane.

We can express the torque $\tau_z(N)$ as a function of the force components X and Y and of the z component $\tau_z(D)$ of the torque exerted by the pendulum on the spring at D:

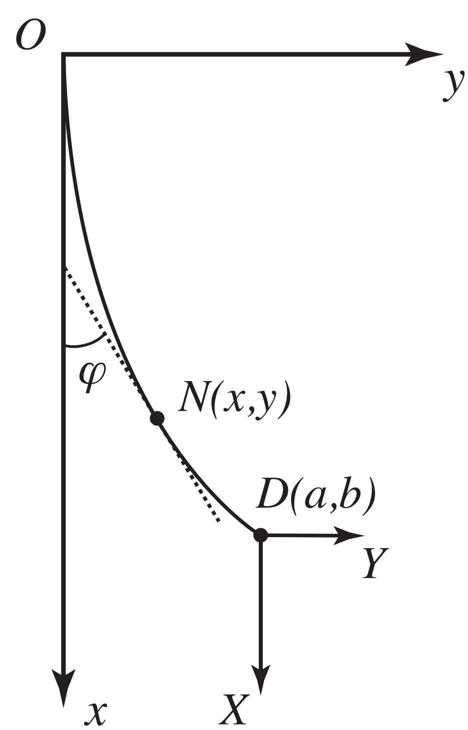

Fig. 2. Schematic of the spring that defines the notation used to analyze the spring shape. The point N (coordinates x, y) lies on the spring and the tangent to the spring at N makes an angle φ with the x-axis. The spring is clamped in the pendulum rod at D (coordinates a, b). The force exerted by the pendulum on the spring at D has components X and Y.

$$\tau_z(N) = (a - x)Y - (b - y)X + \tau_z(D), \tag{6}$$

where (a, b) and (x, y) are the coordinates of points D and N, respectively. The fact that the X and Y force components are independent of the location of point N along the spring is a consequence of our approximation of neglecting the spring inertia.

We are going to calculate the shape of the spring, described by the function $y(x)$. We introduce the arc length s of the curve ON, with $ds = \sqrt{dx^2 + dy^2}$, and the angle φ between the tangent to this curve at N and the x-axis, with $dx/ds = \cos\varphi$ and $dy/ds = \sin\varphi$. With these definitions, the radius of curvature R is given by $1/R = d\varphi/ds$, and Eqs. (4) and (6) then give

$$\mu \frac{d\varphi}{ds} = (a - x)Y - (b - y)X + \tau_z(D). \tag{7}$$

We differentiate Eq. (7) with respect to s in order to obtain an equation involving only the function $\varphi(s)$:

$$\mu \frac{d^2\varphi}{ds^2} = -Y\cos\varphi + X\sin\varphi. \tag{8}$$

This differential equation is difficult to solve, so we simplify it by expanding the right-hand side in powers of φ, retaining only the first-order terms, giving a good approximation for small oscillations:

$$\mu \frac{d^2\varphi}{ds^2} \approx -Y + X\varphi. \tag{9}$$

We will show below that $X = Mg$ so that X is positive. In this case, the solutions of Eq. (9) involve hyperbolic functions:

$$\varphi(s) = \frac{Y}{X} + P\cosh(\kappa s) + Q\sinh(\kappa s), \tag{10}$$

where $\kappa^2 \equiv X/\mu$. Here, P and Q are integration constants, which must be deduced from the conditions at the two ends: at O, where $s = 0$ and $\varphi = 0$, and at D, where $s = l$ and $\mu\, d\varphi/ds = \tau_z(D)$. These initial conditions lead to

$$\begin{aligned} P &= -\frac{Y}{X}, \\ Q &= \frac{\tau_z(D)}{\mu\kappa\cosh\chi} + \frac{Y}{X}\tanh\chi, \end{aligned} \tag{11}$$

where $\chi \equiv \kappa l$ is dimensionless.

Equation (10) gives the spring shape in an implicit form $\varphi(s)$. To get an explicit form, we must integrate the equations $dx/ds = \cos\varphi$ and $dy/ds = \sin\varphi$. As in Eq. (9), we use a first-order expansion in φ to get $dx \approx ds$ and $dy \approx \varphi\, ds$. As $s = 0$ at O (where $x = 0$), integration of the first expression gives $x = s$. Integration of the second expression then gives

$$y(s) = y(x) = \frac{1}{\kappa}[P(\kappa x - \sinh(\kappa x)) + Q(\cosh(\kappa x) - 1)], \tag{12}$$

where we have taken into account that $y = 0$ when $x = 0$. The spring shape is a function of $\chi = \kappa l$ and the ratio P/Q. If $\chi \ll 1$ and if P/Q is not too large, a second-order expansion of Eq. (12) in powers of κx is a good approximation and we find $y \approx Q\kappa x^2/2$, corresponding to a constant radius of curvature. This is the case where the approximation used in Sec. II is valid. When $\chi \gg 1$, the shape is more complex and varies with the ratio P/Q (two examples of the calculated shape are shown in Sec. IV D).

A similar calculation of the spring shape has been performed by James[12] in order to evaluate the stress on the suspension spring and a corrected version of this calculation is reproduced in the book by Matthys.[7] This is a question of great practical interest for clock makers because too large a stress limits the lifetime of the spring.

To study the pendulum dynamics, we need the angle $\theta = \varphi(l)$ and the coordinates a and b of point D. We express these three variables as functions of the force components X and Y and of the torque $\tau_z(D)$. We then substitute P and Q given by Eqs. (11) into Eqs. (10) and (12) to get

$$\begin{aligned} \theta &= A\frac{Y}{X} + B\frac{l\tau_z(D)}{\mu}, \\ a &= l, \\ b &= \frac{Y}{X}l[1 - B] + A\frac{\tau_z(D)}{X}, \end{aligned} \tag{13}$$

where $A \equiv 1 - (1/\cosh\chi)$ and $B \equiv \tanh(\chi)/\chi$ are dimensionless quantities.

B. Second step: Dynamics of the pendulum

Let x_G and y_G be the coordinates of the pendulum center-of-mass G. Newton's equations then give

$$M\frac{d^2x_G}{dt^2} = -X + Mg, \tag{14}$$

$$M\frac{d^2y_G}{dt^2} = -Y, \tag{15}$$

$$M\rho^2\frac{d^2\theta}{dt^2} = -\tau_z(D) + h(Y\cos\theta - X\sin\theta). \tag{16}$$

Next we express x_G and y_G as functions of θ, a, and b:

$$\begin{aligned} x_G &= a + h\cos\theta \approx a + h, \\ y_G &= b + h\sin\theta \approx b + h\theta, \end{aligned} \tag{17}$$

where the approximate values retain only first-order terms in θ. We then rewrite Eqs. (14)–(16) in terms of a, b, and θ. Because x_G is constant, Eq. (14) becomes simply $X = Mg$, and from this we deduce that

$$\kappa = \sqrt{\frac{Mg}{\mu}} = \sqrt{\frac{Mg}{EI_s}}. \tag{18}$$

We now have two equations relating the force component Y and the torque component $\tau_z(D)$ to b and θ, and two differential equations in b and θ:

$$\begin{aligned} &AY + B\kappa^2 l\tau_z(D) = Mg\theta, \\ &[1 - B]lY + A\tau_z(D) = Mgb, \\ &M\frac{d^2b}{dt^2} + Mh\frac{d^2\theta}{dt^2} + Y = 0, \\ &M(\rho^2 + h^2)\frac{d^2\theta}{dt^2} + Mh\left[\frac{d^2b}{dt^2} + g\theta\right] + \tau_z(D) = 0. \end{aligned} \tag{19}$$

With the last two equations, we can express Y and $\tau_z(D)$ as functions of d^2b/dt^2, θ, and $d^2\theta/dt^2$. We introduce these results in the first two equations to get two coupled linear differential equations:

$$\frac{d^2b}{dt^2}\left(A+B\chi^2\frac{h}{l}\right)+\frac{d^2\theta}{dt^2}\left(Ah+B\chi^2\frac{\rho^2+h^2}{l}\right)+\left(1+B\chi^2\frac{h}{l}\right)g\theta=0, \quad (20)$$

$$\frac{d^2b}{dt^2}[(1-B)l+Ah]+\frac{d^2\theta}{dt^2}\left[(1-B)hl+A\left(\rho^2+h^2\right)\right]+g(b+Ah\theta)=0. \quad (21)$$

There are many equivalent techniques for solving these equations. We are looking for harmonic solutions with some angular frequency Ω, so we note that b and θ are both then proportional to $\exp(i\Omega t)$ and we replace d^2/dt^2 by $-\Omega^2$. The consistency of these two homogeneous equations then gives an equation for Ω:

$$U\left(\frac{\Omega^2 l}{g}\right)^2-V\left(\frac{\Omega^2 l}{g}\right)+W=0, \quad (22)$$

where

$$U\equiv\rho^2\left[1-\frac{2\tanh(\chi/2)}{\chi}\right],$$
$$V\equiv h^2+hl+l^2\left(\frac{1}{\chi\tanh\chi}-\frac{1}{\chi^2}\right)+\rho^2, \quad (23)$$
$$W\equiv lh+\frac{l^2}{\chi\tanh\chi}.$$

We use a dimensionless frequency $\Omega^2 l/g$ so that the constants U, V, and W have the same dimension. For each resonance, the associated eigenvector gives the relative amplitudes of b and θ. We define a length $\lambda\equiv b/\theta$ and we deduce λ from Eq. (20):

$$\lambda=\frac{\left(l+B\chi^2h\right)g-\left[Ahl+B\chi^2\left(\rho^2+h^2\right)\right]\Omega^2}{\left(Al+B\chi^2h\right)\Omega^2}. \quad (24)$$

In all cases, the pendulum motion is a rotation around a point of the y-axis, and $(l-\lambda)$ measures the distance of the rotation axis from the origin O.

IV. DISCUSSION OF THE RESULTS

We consider two limiting cases: $\chi\lesssim 1$ and $\chi\gg 1$. We recall that the quantity χ is dimensionless and given by $\chi=\kappa l$, with $\kappa=\sqrt{Mg/(EI_s)}$ (for a clock, a typical χ value is $\chi=2.1$ and, in this case, the spring has only a small effect on the clock frequency). We illustrate our results by considering a typical clock pendulum with a pendulum length $h\approx 1$ m, a radius of gyration $\rho\approx 5$ cm, and a spring length $l\approx 1$ cm, so that $l\ll\rho\ll h$, and we give approximate formulae valid in this case.

A. Approximate solutions of Eq. (22)

The roots of Eq. (22) are

$$\Omega=\pm\sqrt{\frac{g}{l}\frac{V\pm\sqrt{V^2-4UW}}{2U}}. \quad (25)$$

These expressions are useful if one wants to calculate the values of Ω, but the dependence of Ω with the various parameters is not clear. When ρ is small, U is also small, and an expansion of the lowest root of Eq. (22) in powers of U is useful. We define Ω_i as the order-i approximation of the low-frequency root. Then

$$\Omega_0\approx\sqrt{\frac{g}{l}\frac{W}{V}},$$
$$\Omega_1\approx\Omega_0\left(1+\frac{UW}{2V^2}\right), \quad (26)$$

and the high-frequency root is approximately given by

$$\Omega'=\sqrt{\frac{g}{l}\frac{V}{U}}. \quad (27)$$

B. The short-spring case: $\chi\lesssim 1$

For $\chi\lesssim 1$, we expand U, V, and W in powers of χ, which gives $U\approx\rho^2\chi^2/12$, $V\approx h^2+hl+\rho^2+l^2/3$, and $W\approx lh+l^2/\chi^2$. Then Ω_0 is approximately

$$\Omega_0\approx\sqrt{\frac{g(h+l/\chi^2)}{h^2+hl+\rho^2+l^2/3}}, \quad (28)$$

and the first-order correction term $UW/2V^2$ is very small for the typical clock pendulum values. Equation (28) is similar to the approximate result given by Eq. (3) with $l^2/4$ replaced by $l^2/3$; we have no explanation of this small difference but one must not forget that both results are approximate. We are now able to complete the approximate theory of Sec. II by expressing the spring stiffness as $K=-\tau(D)/\theta$. We use Eq. (13) and, because $\chi\ll 1$, we get $A\approx 0$, $B\approx 1$. The spring stiffness is equal to

$$K\approx\frac{\mu}{l}=\frac{EI_s}{l}. \quad (29)$$

The high-frequency root Ω' is given by

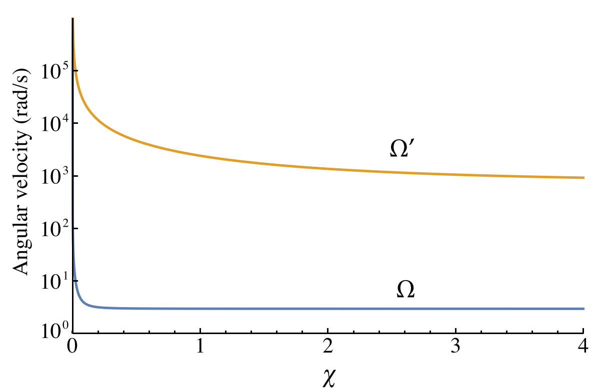

Fig. 3. The values of the low- and high-frequency roots Ω and Ω' of Eq. (22) are plotted as functions of χ for a typical clock pendulum ($h\approx 1$ m, $\rho\approx 5$ cm, and $l\approx 1$ cm). A typical χ value for clocks is close to 2, for which Ω is larger by only about 0.2% compared to if χ were very large.

$$\Omega' \approx \sqrt{12\frac{g}{l}\left(\frac{h^2+\rho^2+hl+l^2/3}{\rho^2\chi^2}\right)}. \tag{30}$$

The ratio $\Omega'/\Omega \approx \sqrt{12h^3/\chi^2\rho^2 l}$ is usually very large; with the typical clock pendulum values, $\Omega'/\Omega \approx 700/\chi$ (see Fig. 3).

C. The long-spring case: $\chi \gg 1$

When $\chi \gg 1$, the functions $\tanh(\chi/2)$ and $\tanh(\chi)$ both tend to 1, and the expansions of U, V, and W up to the first nonzero terms in $1/\chi$ are given by $U \approx \rho^2[1-(2/\chi)]$, $V \approx h^2+hl+\rho^2+l^2/\chi$, and $W \approx hl + l^2/\chi$. With the typical h, ρ, and l values of a clock pendulum, the spring χ value has very small effects on U, V, and W and on the angular frequencies of the two resonances. The low-frequency solution is

$$\Omega_0 \approx \sqrt{\frac{g(h+l/\chi)}{h^2+hl+\rho^2+l^2/\chi}}. \tag{31}$$

The first-order correction term is very small, $UW/2V^2 \ll 1$. The high-frequency root is

$$\Omega' \approx \sqrt{\frac{g}{l}\left[\frac{h^2+hl+\rho^2+l^2/\chi}{\rho^2(1-2/\chi)}\right]} \approx \sqrt{\frac{gh^2}{l\rho^2}}. \tag{32}$$

The ratio $\Omega'/\Omega \approx \sqrt{h^3/(\rho^2 l)}$ is very large: $\Omega'/\Omega \approx 200$ with the typical l, h, and ρ values; the high-frequency oscillation is usually difficult to observe.

We note that it is possible to considerably reduce the Ω'/Ω ratio by a careful choice of the values of l, h, and ρ. The Ω'/Ω ratio is lowest when the ratio $4UW/V^2$ is maximum. Moreover, as we assume $\chi \gg 1$, we may neglect the χ^{-1} terms and then $4UW/V^2$ is a function solely of l, h, and ρ. Rather than attempting a global optimization, we consider only the realistic case of a pendulum with a body made of a small-diameter bar of length $2h$. Then $\rho^2 \approx h^2/3$, and the maximum of $4UW/V^2$, obtained for $l=4h/3$, is equal to $4UW/V^2 = 1/4$, which leads to a ratio $\Omega'/\Omega \approx 3.7$. With such a low value of this ratio, the high-frequency motion should be easy to observe. We have recently built such a pendulum and we have measured the frequencies of the two resonances as a function of the length l, with preliminary results in satisfactory agreement with the present theory.

D. The spring shape

Equation (24) suggests that λ is positive for the low-frequency resonance and negative for the high-frequency resonance. We have numerically verified this property over a wide range of parameters. The spring shape for the two motions is shown for a particular case in Fig. 4.

E. Comparison to previous results

We now discuss the papers of Hughes[8] and of Gleiser,[9] which are connected to our work.

Hughes[8] considered a pendulum made of a sphere suspended by an infinitely soft string. He found two resonances: one for the usual pendular motion and one for a rolling motion in which the sphere oscillates about an axis very near its center. For a soft string, the quantity $\mu = EI_s$ goes to zero,

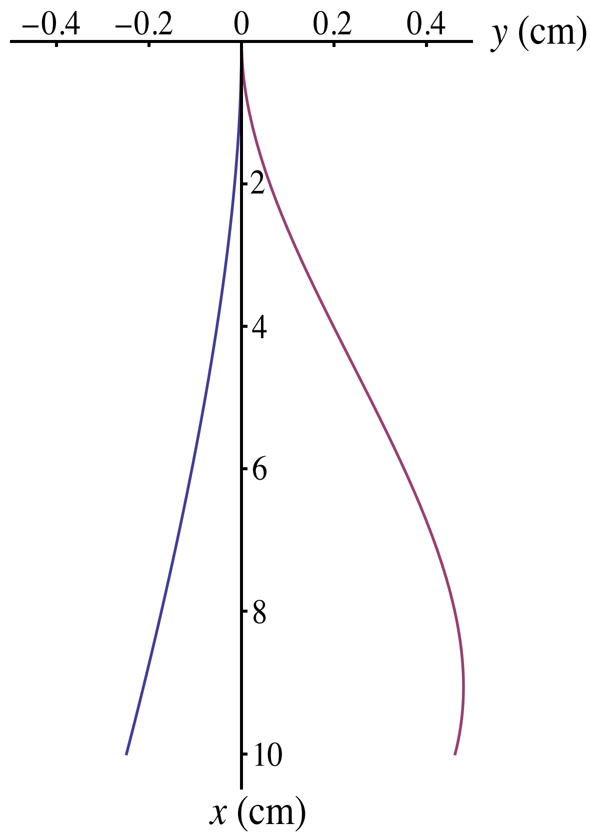

Fig. 4. Spring shape for the low-frequency resonance ($\Omega = 60$ rad/s) on the left and the high-frequency resonance ($\Omega' = 3\,800$ rad/s) on the right. The calculation is carried out using $\chi = 2.1$, $l = 10$ cm, $h = 10$ cm, and $\rho = 5$ cm.

so that $\kappa = \sqrt{Mg/\mu}$ and $\chi = \kappa l$ both tend toward infinity. We have verified that, in the limit $\chi \to \infty$, Eq. (22) is equivalent to Eq. (5) of Hughes.[8]

Gleiser[9] used a variational formalism to describe a pendulum with arbitrary elastic properties and mass distribution. He found that the value of the angular frequency is independent of the flexural rigidity at first order in this quantity, denoted as κ in his paper and equal to our μ. This surprising result can be easily explained. When μ tends to 0, $\chi \to \infty$ and the coefficients U, V, and W of Eq. (22) are of the form $C + D\chi^{-1}$, where C and D are constants. The low-frequency root Ω has a similar form $(C' + D'\chi^{-1})$ and, as $\chi \propto 1/\sqrt{\mu}$, $\Omega = C' + D''\sqrt{\mu}$. This non-analytic behavior of Ω as a function of μ is sufficient to explain the failure of a simple variational calculation. We may even remark that the variational solution used by Gleiser to describe the shape of the spring is a straight line, from point O to point D in our notation. This form is correct if $\mu = 0$, but for any nonzero value of μ, the real shape of the spring must be tangent to the vertical axis at O, a property not verified by his test solution.

We have calculated the shape of the spring for the pendular motion in the limit of a soft spring ($\chi \gg 1$), and get

$$\varphi = \theta(1-e^{-\kappa x}), \tag{33}$$

$$y(x) = \theta\left(x - \frac{1-e^{-\kappa x}}{\kappa}\right). \tag{34}$$

Equation (33) shows that the angle φ between the tangent to the spring and the x-axis varies from 0 to its terminal value θ over a distance comparable to κ^{-1}. This means that the curvature of the spring is localized near the origin O, within a distance that goes to 0 as $\sqrt{\mu}$.

V. CONCLUSION

In this paper, we have given a more complete treatment of a pendulum suspended by a spring. The spring shape is described by the Euler-Bernoulli equation, and our treatment is rigorous in the limit of small oscillations. Our calculation predicts two resonances, one corresponding to the pendular motion at low frequency and the other to a rolling motion at high frequency. This result can be understood as a particular case of a double pendulum, as already discussed by Hughes,[8] where the spring is replaced by an infinitely soft string.

We have considered mainly the case of a typical clock pendulum, where the suspension spring only slightly modifies the angular frequency of the low-frequency resonance associated to the usual pendular motion. The high-frequency resonance occurs in that case at a considerably larger frequency, so it is not easily observed. However, we have shown that with a careful choice of the pendulum dimensions, the ratio of these two frequencies can be considerably reduced, and we have verified experimentally that both frequencies can then be easily observed.

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[a]Electronic mail: gilles.dolfo@irsamc.ups-tlse.fr

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This **Movable Planisphere of the Heavens at Every Minute**, a cardboard aid to astronomy students, was published by Henry Whitall, 512 Arch Street, Philadelphia, Pennsylvania in 1862. The sentence "To Astronomy what a Map is to a Geographer" is at the center of the instrument and the Greek alphabet, "as applied to the stars", is given for reference on the left-hand side. The revolving part shows the constellations. The description in the 1889 Queen catalogue is: "Whitall's Planisphere. Showing the stars and constellations visible at any hour in the evening, for every night of the year. The most simple and satisfactory map of the heavens for the use of students and others extant. On strong cardboard. 15 inches in diameter. ...$3.00" Queen also sold a version on glass for use in a projector. This example is at Washington and Lee University. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College).