

Phase-space manipulations of many-body wave functions

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We explore the manipulation in phase space of many-body wave functions that exhibit self-similar dynamics under the application of sudden force and/or in the presence of a constant acceleration field. For this purpose, we work out a common theoretical framework based on the Wigner function. We discuss squeezing in position space, phase-space rotation, and its implications in cooling for both noninteracting and interacting gases and time-reversal operation. We discuss various optical analogies and calculate the role of a spherical-like aberration in cooling protocols. We also present the equivalent of a spin-echo technique to improve the robustness of velocity dispersion reduction protocols.

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I. INTRODUCTION

Phase-space manipulations are at the heart of astonishing developments in atomic and molecular physics. Laser cooling of cold-atom samples provides a spectacular example of the increase of the phase-space density by populating a large number of photon modes through the dissipative mechanism provided by spontaneous emission [1]. Alternatively, evaporative cooling exploits the irreversible nature of three-dimensional elastic collisions to increase the phase-space density of a subensemble of confined particles. The phase-space density of atomic beams has also been increased with similar techniques [2,3]. The demonstration of Maxwell's demon devices that combined conservative potentials with an irreversible step belongs to the same kind of phase-space manipulations [4,5]. In the absence of dissipative mechanisms, the phase-space volume is conserved. The manipulations that can be carried out in phase space with well-engineered time-dependent conservative potentials involve, separately or in combination, translation (used, for instance, in the slowing down of atomic or molecular packets [6–10]) and deformation [7,11,12], including compressions either in position or momentum space [11,13–18] or magnification [19,20]. Such methods are quite general; they do not rely on a specific internal structure and can therefore be applied to a large class of particles, including neutrons [21,22].

In Ref. [14], the authors proposed a phase-space manipulation for velocity-dispersion reduction of a noninteracting wave packet based on phase imprinting. This method has proven to cool very efficiently thermal and Bose-condensed atomic samples, leading recently to temperatures as low as 50 pK [23]. This efficient narrowing of the velocity dispersion is of great interest in metrology measurements based on atom interferometry [24,25] and also for realizing quantum simulations [26].

Very recently, such techniques have started to be applied to strongly interacting atoms [19]. In this paper, we precisely investigate the generalization of such a cooling concept for manipulating, in phase space, many-body quantum systems that exhibit self-similar dynamics [27–38].

This paper is arranged as follows. The scaling formalism applied to the Wigner function and the class of many-body systems for which it is valid are presented in Sec. II. Section III

derives the maximum compression factor in space that one can obtain for a given kick force depending on the evolution law of the dilation factor. In Sec. IV, we provide the Wigner formalism for a general compression and displacement in momentum space. We also show how a time-reversal operator can be applied. Section V provides a concrete comparison between noninteracting and interacting cases. The issue of anharmonicities is investigated in Sec. VI. The last section explores more involved phase-space manipulations for improving the robustness of compression protocols.

II. WIGNER FUNCTION FOR SELF-SIMILAR MANY-BODY SYSTEMS

The many-body quantum systems that exhibit self-similar dynamics include the Calogero-Sutherland model [29], the Tonks-Girardeau gas [30,31], certain Lieb-Liniger states [39], Bose-Einstein condensates [32,33] even in the presence of dipolar interactions [34], strongly interacting gas mixtures [35], strongly interacting quantum gases whose collisions are described by the unitary limit [36], etc. A noninteracting classical gas described by its phase-space distribution function governed by the Boltzmann equation also belongs to the same class of problems [37].

The formalism that we use relies on the Wigner function W associated with the many-body wave function. It is defined via the one-body reduced density matrix $g_1(x, y; t)$:

$$W(x, p; t) = \frac{1}{\pi \hbar} \int g_1(x + y, x - y; t) e^{2ipy/\hbar} dy. \quad (1)$$

The self-similar dynamics after phase imprinting or in the presence of a constant acceleration field g involve two time-dependent parameters, $\alpha(t)$ and $\eta(t)$ [40]:

$$g_1(x, y; t) = \frac{1}{\alpha} g_1\left(\frac{x - \eta}{\alpha}, \frac{y - \eta}{\alpha}; 0\right) e^{i[S(x,t) - S(y,t)]}, \quad (2)$$

where the time-dependent dilation factor fulfills $\ddot{\alpha} = \omega_0^2/\alpha^\xi$ (the exponent ξ depends on the specific system that is considered) for a free propagation of a many-body wave function initially at rest in a one-dimensional (1D) harmonic confinement of angular frequency ω_0 . The time-dependent function η accounts for the center-of-mass motion $\ddot{\eta} = g$, where g is a constant acceleration field. The phase $S(x, t)$

is given by

$$S(x,t) = \frac{m\alpha}{\hbar} \left[\dot{\eta} \left(\frac{x-\eta}{\alpha} \right) + \frac{\dot{\alpha}}{2\alpha} \left(\frac{x-\eta}{\alpha} \right)^2 \right]. \quad (3)$$

As a result of the self-similar dynamics, the instantaneous Wigner function $W(x,p;t)$ is simply related to the initial Wigner $W_0(x,p)$ function through the relation $W(x,p;t) = W_0(X,P)$, with $X = (x-\eta)/\alpha$ and $P = \alpha(p - m\dot{\eta}) - m\dot{\alpha}(x-\eta)$ [38,40].

Without loss of generality and for pedagogical reasons, in the following we shall compare quite often two specific cases of a pure state: (i) a Gaussian wave packet without interactions ($\xi = 3$) and (ii) the mean-field wave function associated with a Bose-Einstein condensate (BEC) in the Thomas-Fermi regime ($\xi = 2$). In both cases, the initial wave function reads $\Psi(x,0) = n_0^{1/2}(x)$, where n_0 is the atomic density. In the former case, the Gaussian wave function reads $n_0(x) = e^{-x^2/\sigma_0^2}/(\pi^{1/2}\sigma_0)$ and can be considered to be the ground state of a harmonic trap of angular frequency ω_0 [$\sigma_0 = (\hbar/m\omega_0)^{1/2}$]. In the latter case of a Bose-Einstein condensate in the Thomas-Fermi regime, the density in the same harmonic potential reads $n_0(x) = (\mu - m\omega_0^2 x^2/2)/\tilde{g} = m\omega_0^2(R_{\text{TF}}^2 - x^2)/(2\tilde{g})$, where μ is the chemical potential, \tilde{g} is the strength of the interactions, and $R_{\text{TF}} = (2\mu/m\omega_0^2)^{1/2}$ is the Thomas-Fermi radius.

III. COMPRESSION IN POSITION SPACE

The simplest phase-space manipulation is the position squeezing of the wave packet (see Fig. 1). The first strategy consists of increasing the trap strength adiabatically to the desired value. Alternatively, one can use a much faster approach by suddenly applying a linear force, $F = -m\omega^2 x$, for a very short amount of time Δt . This action amounts to setting the time derivative of the dilation factor α , $\dot{\alpha}(0^+) = -\omega^2 \Delta t$. Such a force can be realized with either a pulsed magnetic or optical trap. From the differential equation fulfilled by the time-dependent parameter α , we deduce an energylike constant of motion:

$$\frac{d}{dt} \left[\left(\frac{d\alpha}{dt} \right)^2 + \omega_0^2 \frac{\alpha^{-\xi+1}}{1-\xi} \right] = 0. \quad (4)$$

This equation is particularly well suited for determining the minimum value of the dilation factor α_{min} resulting from the kick force:

$$\alpha_{\text{min}}(\xi; \omega, \Delta t) = \left(1 + (1-\xi) \frac{\omega^2}{\omega_0^2} (\omega \Delta t)^2 \right)^{1/(1-\xi)}. \quad (5)$$

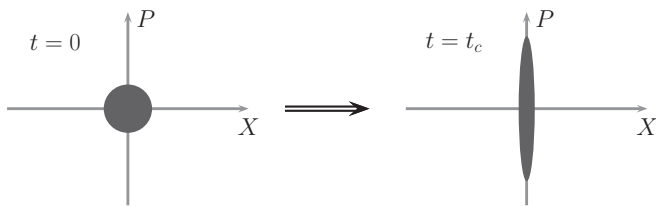


FIG. 1. Focalization of a wave packet by suddenly applying an attractive harmonic force in a very short amount of time.

The wave packet reaches its minimum size, $(\Delta r)_c = \alpha_{\text{min}}(\Delta r)_0$, at time $t_c \sim 1/\omega_0$, where $(\Delta r)_0$ is its initial size. By conservation of the phase-space volume, the velocity dispersion at that time is increased by the same factor $(\Delta v)_c = (\Delta v)_0/\alpha_{\text{min}}$. As intuitively expected, we find $\alpha_{\text{min}}(3; \omega, \Delta t) < \alpha_{\text{min}}(2; \omega, \Delta t)$. Indeed, the repulsive interactions encapsulated in the exponent $\xi = 2$ for a BEC in the Thomas-Fermi regime limit the compressibility of the gas compared to its interaction-free counterpart ($\xi = 3$). To ensure that the force remains sudden, one has to fulfill the condition $\omega \Delta t \ll 1$. At its minimum size, the wave function exactly coincides with that of the ground state of a harmonic oscillator potential of angular frequency $\omega_c = \omega_0[\alpha_{\text{min}}(\xi; \omega, \Delta t)]^{1/(1-\xi)}$; that is, if we suddenly apply at t_c such a harmonic potential, the packet remains unchanged because of the size matching and the phase cancellation, $S(x, t_c) = 0$.

IV. COMPRESSION IN MOMENTUM SPACE

To reduce the velocity dispersion, one could progressively decrease the trap strength. However, the lower the final angular frequency is, the larger the time required to ensure the adiabaticity criterion is. Furthermore, the sensitivity to low-frequency noise in experimental setups ultimately limits the achievable velocity dispersion. To circumvent those limitations we investigate a two-step protocol initially studied for the interaction-free wave packet and commonly referred to as the δ -kick cooling protocol [14].

We detail this protocol for a pure state. The generalization to the self-similar many-body wave function is straightforward [27]. In such a protocol, the wave function first expands freely under a constant acceleration field g during a time interval t_- . Figure 2(a) represents an example of such an evolution in the case of a free expansion ($g = 0$). Then at time $t = t_-$, a force $F = -m\Omega^2(x - x_0)$ is applied suddenly and for a very short amount of time (short with respect to all other time scales), where x_0 accounts for a position offset between the center of the applied harmonic potential and the center of the atomic cloud. As a result, the wave function acquires an extra phase factor just after the application of the force (at time t_+ such that $|t_+ - t_-| \ll t_-$)

$$\psi(x, t_+) = \exp\{-i\gamma(x - x_0)^2\} \psi(x, t_-),$$

where $\gamma = m\Omega^2(t_+ - t_-)/(2\hbar)$. Applying this force amounts to rotating the phase-space volume [see Fig. 2(b)]. The corresponding Wigner function reads

$$W(x, p, t_+) = \frac{1}{N\pi\hbar} \int dY f_0(X, Y) e^{2iPY/\hbar},$$

with $P = \alpha(p + 2\hbar\gamma(x - x_0) - m\dot{\eta} - m\dot{\alpha}X)$, $f_0(X, Y) = n_0^{1/2}(X+Y)n_0^{1/2}(X-Y)$, and $Y = y/\alpha$. From this expression, we infer the mean value of the momentum

$$\begin{aligned} \langle p \rangle &= \int dx dp p W(x, p, t_+) \\ &= \frac{1}{N\pi\hbar} \int dXdP \left[\frac{P}{\alpha} + aX + p_t \right] f_0(X, Y) e^{2iPY/\hbar} \\ &= \frac{\langle p \rangle_0}{\alpha} + a\langle x \rangle_0 + p_t, \end{aligned} \quad (6)$$

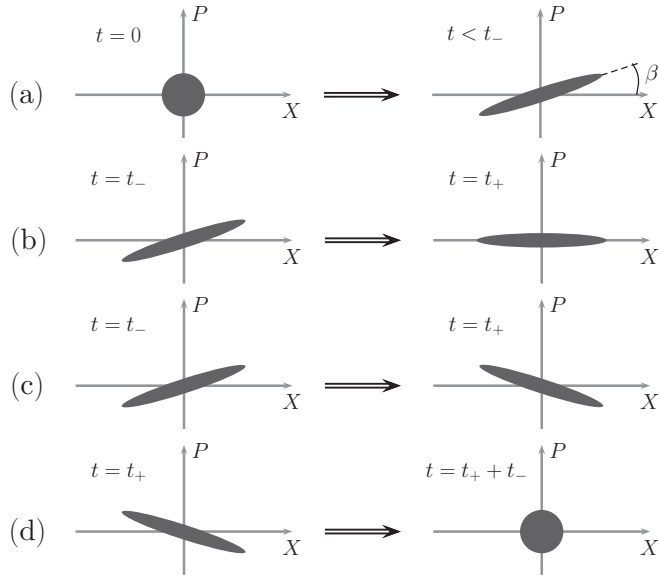


FIG. 2. To squeeze the wave packet along the P axis, two successive steps are considered: (a) a free expansion, (b) followed by a δ -kick force with the right strength to ensure the rotation by the appropriate angle β^* . (c) A rotation by twice this angle, simply obtained by an applied force two times larger than that used for the squeezing, reverses the position-velocity correlation and therefore acts as a time-reversal operator. The complete time-reversal process implies three steps: (a) free propagation during a finite time τ , (c) rotation by an angle $2\beta^*$, and (d) free propagation over a duration equal to τ .

with $a = m\dot{\alpha} - 2\hbar\gamma\alpha$ and

$$p_t = m\dot{\eta} - 2\hbar\gamma(\eta - x_0). \quad (7)$$

The quantity p_t provides an offset in momentum space. A similar calculation for $\langle p^2 \rangle$ yields

$$\begin{aligned} \langle p^2 \rangle &= \frac{\langle p^2 \rangle_0}{\alpha^2} + a^2 \langle x^2 \rangle_0 + p_t^2 \\ &+ \frac{2p_t}{\alpha} \langle p \rangle_0 + 2ap_t \langle x \rangle_0 + \frac{2a}{\alpha} \langle xp \rangle_0. \end{aligned}$$

We conclude that after the two-step protocol, the variance of the momentum reads

$$(\Delta p)^2 = \frac{(\Delta p)_0^2}{\alpha^2} + a^2 (\Delta x)_0^2 + \frac{2a}{\alpha} (\langle xp \rangle_0 - \langle x \rangle_0 \langle p \rangle_0).$$

The optimal choice to minimize the momentum dispersion is $a = 0$, i.e., $m\dot{\alpha} = 2\hbar\gamma\alpha$. It corresponds to the optimal rotation angle $\beta = \beta^*$ illustrated in Fig. 2(b). The value to be used for γ therefore depends on the evolution of the dilation factor α , and its strength is experimentally tunable by the control of the quantity $\Omega^2(t_+ - t_-)$. The corresponding velocity dispersion reduction is $\Delta v(t_+) = \Delta v(0)/\alpha_-$, with $\alpha_- = \alpha(t_-)$. This result is particularly interesting since it yields a reduction in the velocity dispersion with respect to the *initial* one in a short amount of time and therefore shortcuts any adiabatic transformation [41]. In the absence of interactions, we find $\alpha(t) = (1 + \omega_0^2 t^2)^{1/2}$, which yields $\Delta v(t) = \Delta v(0)/(1 + \omega_0^2 t^2)^{1/2}$. The specific angle value, $\beta = \beta^*$, is also the one that cancels the quadratic term in x

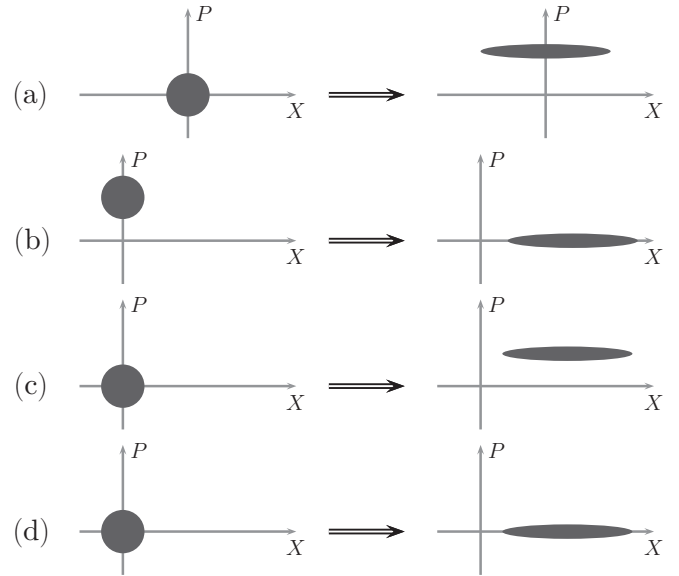


FIG. 3. (a) and (b) Using a linear kick force with an offset, the cooling and translation on the momentum axis can be obtained simultaneously. (c) and (d) The final position of the squeezed wave packet can be chosen in phase space by an appropriate sequence of evolution under a constant force followed by the application of a sudden force with or without offset.

in the phase $S(x, t_+)$. As a result, the wave function is, at t_+ , in the form $\Psi(x, t_+) = \sqrt{n_0(x/\alpha_-)/\alpha_-^{1/2}} \exp(i\kappa x)$ with a time-dependent global phase factor.

The linear term in x in the phase accounts for the mean momentum $\hbar\kappa$ that can be acquired in the two-step protocol and/or depends on the initial conditions. For the optimal choice $\beta = \beta^*$, the mean momentum is equal to $\langle p \rangle = \langle p \rangle_0/\alpha + p_t$. After the two-step protocol, the momentum is therefore translated by a quantity $\langle p \rangle - \langle p \rangle_0 = \langle p \rangle_0(\alpha^{-1} - 1) + p_t$. According to Eq. (7), there are two independent ways to communicate a mean momentum to the packet while reducing the momentum dispersion: either using a constant acceleration field ($g \neq 0$) or applying an off-center sudden harmonic potential ($x_0 \neq 0$). These two possibilities can also be combined [see Eq. (7)]. In Fig. 3, a few examples of possible phase-space manipulations are described. Using a harmonic potential with an offset ($x_0 \neq 0$) for the phase imprinting, one can simultaneously launch a packet and cool it [see Fig. 3(a)]. Using an order of magnitude with realistic values for a noninteracting wave packet with $\omega_0 = 2\pi \times 150$ Hz and a free-expansion time of 5 ms, we find $\dot{\alpha}/\alpha = 192$ s $^{-1}$; for an offset of $x_0 = 20$ μ m, we get $\bar{v} = 3.8$ mm/s.

The application of a sudden force imprints a quadratic with position phase and is thus in close analogy with the action in the Fourier space of a lens in optics. The role of the standard δ -kick cooling protocol is to remove the phase acquired during the propagation. The velocity resulting from the off-center δ -kick cooling corresponds to the use of two lenses with opposite focal lengths f but with a displacement of their centers by a quantity a . The optical analog corresponds to an optical beam arriving from infinity and deflected by an angle $-a/f$. This angle plays the same role as the offset mean velocity.

We have already seen that the choice of β that provides the optimal rotation amounts to canceling the quadratic term of the phase of the wave function. In the absence of center-of-mass motion, if one rotates the phase space distribution by twice this angle, $\beta = 2\beta^*$, the phase is reversed: $S(x, t_+) = -S(x, t_-)$. As a result, $\Psi(x, t_+) = \Psi^*(x, t_-)$, an operation that corresponds exactly to the time-reversal operator. This means that the wave function will reconcentrate towards its original form despite the repulsive interactions if any. This latter situation is illustrated in Fig. 2(c). The free evolution is then a refocusing towards the initial state [see Fig. 2(d)]. Actually, this time-reversal technique enables one to determine precisely the gain on velocity reduction. Indeed, the value may be so low that the standard time-of-flight technique can no longer be used in standard setups. In this case, the time-reversal operation provides a way to infer the velocity dispersion obtained for a rotation by the angle β^* in the δ -kick cooling protocol. We shall see another application of this time-reversal possibility in Sec. VII.

V. QUANTITATIVE COMPARISON BETWEEN THE NONINTERACTING AND INTERACTING CASES

The initial velocity dispersion for a trapped Bose-Einstein condensate in the Thomas-Fermi regime is small compared to its noninteracting counterpart since the size of the atomic sample is larger. However, the velocity dispersion increases when the confinement is removed due to the conversion of the interaction energy into kinetic energy.

In the following, we propose a quantitative comparison of the relative performances of the two-step protocol for the interaction-free Gaussian wave function and the wave function of a BEC in the Thomas-Fermi (TF) regime. For the sake of simplicity, we restrict ourselves to the situation in which the wave packet is initially at rest and experiences no constant acceleration field ($g = 0$). The generalization to $g \neq 0$ of the solution worked out hereafter is straightforward.

Let us first recall the expression for the velocity dispersion of a BEC in the TF regime [42]:

$$(\Delta v_{\text{TF}})(0) = \Delta v_0 \frac{\sqrt{2}}{\delta} \sqrt{\ln(\theta\delta)}, \quad (8)$$

with $\theta \simeq 1.373475 \dots$ being a numerical factor, $\delta = R_{\text{TF}}/\sigma_0$, and $\Delta v_0 = (\hbar\omega_0/2m)^{1/2}$ being the velocity dispersion for the noninteracting Gaussian wave function. The optimal rotation angle β^* defined by $a = 0$ takes into account the deformation of the phase-space surface induced by the interactions through the exponent ξ in the evolution of the dilation factor α . For $t < t_-$ the velocity dispersion is given by [40]

$$(\Delta v)(t) = \Delta v_{\text{TF}}(0) \left[\frac{1}{\alpha(t)^2} + \left(1 - \frac{1}{\alpha(t)}\right) \frac{\delta^2}{\ln(\theta\delta)} \right]^{1/2}. \quad (9)$$

At $t = t_+$, $\Delta v(t_+) = \Delta v_{\text{TF}}(0)/\alpha_-$. Figure 4 gives the evolution of the ratio $(\Delta v)(t_+)/(\Delta v_0)$ as a function of time for different values of the ratio $\delta = R_{\text{TF}}/\sigma_0$ just after the two-step protocol is applied. At first glance, one could conclude that there is a very impressive gain in terms of the reduction of the velocity dispersion in applying the δ -kick cooling technique on a many-body wave function.

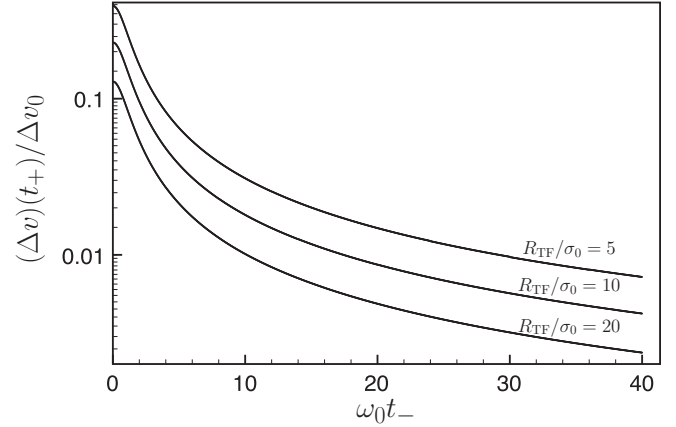


FIG. 4. Velocity dispersion just after the δ -kick cooling protocol as a function of the time t_- at which the rotation is performed and for various values of the ratio between the initial Thomas-Fermi radius R_{TF} and the harmonic length σ_0 . The velocity dispersion is normalized to that of the Gaussian ground-state wave function in the same initial harmonic confinement, and the time is normalized to ω_0^{-1} .

However, a more rigorous comparison requires the study of the evolution after the two-step protocol, i.e., $t > t_+$. Indeed, the interaction energy remains as a reservoir that can significantly increase the velocity dispersion. From an analytical perspective, the answer turns out to be not that simple despite the fact that we know exactly the expression for the wave function at t_+ : $\Psi(x, t_+) = \sqrt{\tilde{n}(x)}$, where $\tilde{n}(x) = n_0(x/\alpha_-)/\alpha_-^{1/2}$, with $\alpha_- = \alpha(t_-)$. Indeed, this inverse parabola shape corresponds to that of a Bose-Einstein condensate in the Thomas-Fermi regime for which the interaction strength would be $g_+ = \tilde{g}/\alpha_-^2$ and the confinement strength would also be reduced, $\omega_+ = \omega_0/\alpha_-$. The 1D dimensionless χ parameter given by the ratio of the interaction energy over the kinetic energy is thereby drastically reduced, $\chi_+ = \chi/\alpha_-^{5/2}$. For a standard expansion (by a factor of 3–10), this reduction factor is large, and the wave function can no longer be considered to be that of a Bose-Einstein condensate in the Thomas-Fermi regime. The dilution is too important, and the interaction energy is no longer important compared to the kinetic energy. As a result, a scaling ansatz can no longer be used to account for the time evolution of the wave function at $t > t_+$; only numerics can provide the exact evolution [43]. Alternatively, one can analytically obtain the asymptotic value of the velocity dispersion. For this purpose, we need to calculate the kinetic energy $E_{\text{kin}}(t_+)$ and the interaction energy $E_{\text{int}}(t_+)$, and we shall use the conservation of energy:

$$(\Delta v)^2(\infty) = \frac{2}{mN} [E_{\text{kin}}(t_+) + E_{\text{int}}(t_+)]. \quad (10)$$

We can directly calculate the interaction energy: $E_{\text{int}}(t_+) = E_{\text{int}}(0)/\alpha_-$ and $E_{\text{int}}(0) = N\hbar\omega_0\delta^2/5$. The expression for the kinetic energy can be obtained from Ref. [40],

$$E_{\text{kin}}(t_+) = \frac{N\hbar\omega_0}{2\alpha_-^2\delta^2} \ln(\theta\delta).$$

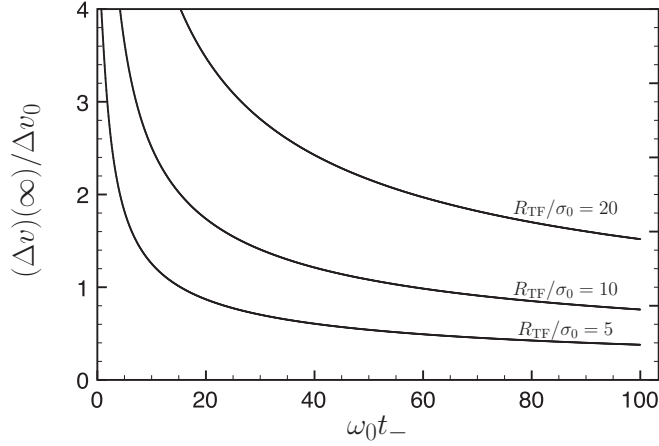


FIG. 5. Asymptotic velocity dispersion as a function of the time t_- at which the rotation is performed and for various values of the ratio between the initial Thomas-Fermi radius R_{TF} and the harmonic length σ_0 . Same normalization as in Fig. 4.

From Eq. (10), we infer the asymptotic velocity dispersion for $t \gg t_+$,

$$(\Delta v)(\infty) = \Delta v_0 \left[\frac{4\delta^2}{5\alpha_-} + \frac{2\ln(\theta\delta)}{\alpha_-^2 \delta^2} \right]^{1/2}.$$

In Fig. 5, the evolution of the ratio $\Delta v(\infty)/\Delta v_0$ as a function of time for different values of the ratio $\delta = R_{TF}/\sigma_0$ is plotted. We obtain a much more moderate gain. Interestingly, this ratio always goes below 1 if the phase imprinting is applied after a sufficiently long free-expansion time, meaning that the contribution of repulsive interactions to the velocity dispersion can be completely washed out using the two-step protocol and that a dispersion below the noninteracting case can even be achieved. It is worth noticing that this latter prediction is the most pessimistic one. Indeed, the time required to reach this asymptotic value increases considerably (by at least a factor of α_-) after the phase-space rotation because of the dilution. For an experiment carried out on a time scale that is short compared to α_-/ω_0 , one can really benefit from the gain in velocity dispersion presented in Fig. 4.

VI. ABERRATION

In this section, we investigate the influence of anharmonicities in the suddenly applied potential for the δ -kick cooling technique. Calculations are carried out for a pure state initially at rest ($\eta = 0$). We consider that the phase imprinting does not correspond exactly to a quadratic force but contains an extra cubic correction. We propose a perturbative analysis of this effect that would correspond to spherical aberration in optics. Under our assumption, the wave function after the phase imprinting reads

$$\begin{aligned} \psi(x, t_+) &= e^{-i\gamma x^2 + i\epsilon x^3} \psi(x, t_-) \\ &\simeq e^{-i\gamma x^2} \psi(x, t_-) \left(1 + i\epsilon x^3 - \frac{\epsilon^2 x^6}{2} \right). \end{aligned} \quad (11)$$

The Wigner function contains an extra factor $\Lambda(x, y)$ compared to that in the case without anharmonicities:

$$\begin{aligned} W(x, p, t_+) &= \frac{1}{N\pi\hbar} \int \psi^*(x+y, t_+) \psi(x-y, t_+) e^{2ipy/\hbar} dy \\ &= \frac{1}{N\pi\hbar} \int \psi^*(x+y, t_-) \psi(x-y, t_-) \\ &\quad \times \Lambda(x, y) e^{2i\tilde{p}y/\hbar} dy, \end{aligned} \quad (12)$$

with $\tilde{p} = p + 2\hbar\gamma x$. The expansion up to the second order in ϵ of Λ reads $\Lambda(x, y) \simeq 1 - 2i\epsilon y(3x^2 + y^2) - 2\epsilon^2 y^2(9x^4 + 6x^2 y^2 + y^4)$. We deduce

$$W(x, p, t_+) = \frac{1}{N\pi\hbar} \int dY g_0(X, Y) e^{2iPY/\hbar} dy,$$

with $g_0(X, Y) = f_0(X, Y)\Lambda(\alpha X, \alpha Y)$, $P = \alpha\tilde{p} - m\dot{\alpha}x$, $X = x/\alpha$, $Y = y/\alpha$. Using the same notation as before, we get

$$\begin{aligned} \langle p \rangle &= \int dx dp p W(x, p, t_+) \\ &= \frac{1}{N\pi\hbar} \int dX dP \left[\frac{P}{\alpha} + \alpha X \right] g_0(X, Y) e^{2iPY/\hbar} \\ &= \frac{1}{N\pi\hbar} \frac{1}{\alpha} \int dX \left(-\frac{\pi\hbar^2}{2i} \right) \partial_Y g_0(X, Y) \Big|_{Y=0} + 0 \\ &= \frac{1}{N\pi\hbar} \frac{1}{\alpha} \left(-\frac{\pi\hbar^2}{2i} \right) (-6i\epsilon\alpha^3) \int X^2 \psi_0(X) dX \\ &= 3\epsilon\alpha^2 \hbar \langle x^2 \rangle_0. \end{aligned} \quad (13)$$

A similar calculation for $\langle p^2 \rangle$ yields $\langle p^2 \rangle = \langle p^2 \rangle_0/\alpha^2 + \alpha^2 \langle x^2 \rangle_0 + 9\epsilon^2 \alpha^4 \hbar^2 \langle x^4 \rangle_0$. We conclude that the optimal variance of the momentum is given by

$$(\Delta p)^2 = (\Delta p)_0^2/\alpha^2 + (\Delta p)_\epsilon^2,$$

where $(\Delta p)_\epsilon^2 = 9\epsilon^2 \alpha^4 \hbar^2 (\langle x^4 \rangle_0 - \langle x^2 \rangle_0^2)$. We conclude that the optimal angle remains the same as in the case without aberration, but the reduction of the velocity dispersion is clearly affected by the nonlinearity. The effect is particularly strong in the regime in which the two terms are on the same order of magnitude, i.e., for $t > t_\epsilon$ with t_ϵ defined by $\alpha(t_\epsilon) = (\Delta p)_0/(\Delta p)_\epsilon$.

VII. HIGHER-ORDER PROTOCOLS

In this section, we design an ultrarobust many-step protocol inspired by the spin-echo technique well known in NMR [44]. This technique, used for robust quantum information processing, allows for an improved fidelity of a π -pulse operation in the presence of dispersion of Rabi frequencies. Its simplest version relies on a sequence of three successive pulses: $(\pi/2)_Y(\pi)_X(\pi/2)_Y$. After the first pulse the spin directions are spread over a finite angle of about $\pi/2$. The spins with the largest Rabi frequency have rotated by more than $\pi/2$. The second pulse reverses on the Bloch sphere the relative positions of the slowest and fastest Rabi frequency spins. Finally, the last pulse refocuses all spins.

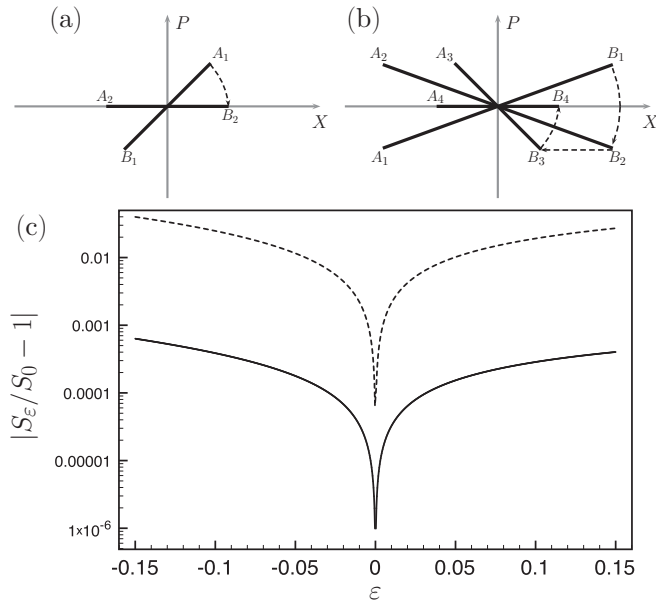


FIG. 6. (a) Strategy 1: simple phase-space rotation (dashed line). (b) Strategy 2: spin-echo-like phase-space manipulation (solid line). For the sake of simplicity, we only represent the long axis of the ellipse shape of the phase-space volume. (c) The relative error on the phase $|S_\varepsilon/S_0 - 1|$ as a function of the frequency mismatch ε .

The transposition of this technique in phase-space manipulation is summarized in Fig. 6. We compare the robustness against fluctuations of the initial trap frequency [$\omega = \omega_0(1 + \varepsilon)$] of two different schemes aimed at reducing the velocity dispersion. This provides a way to probe the robustness against anharmonicities. The first one is the standard δ -kick cooling strategy [see Fig. 6(a)]; it consists of free propagation

over a time T followed by a sudden clockwise optimal phase-space rotation. The second one involves free propagation over a time T' , a sudden time-reversal force pulse, free propagation over a time T'' , and a sudden counterclockwise rotation [see Fig. 6(b)]. The time interval T' is chosen so that $\alpha(T') = 2\alpha(T)$, and the time T'' is chosen so that the final state is the same as in the first strategy. The time-reversal pulse plays the same role as the $(\pi)_X$ pulse in the spin-echo protocol: it reverses the position of the slowest and fastest atoms, i.e., the atoms that initially experience a smaller or larger trap frequency. The final counterclockwise pulse refocuses the different trajectories. The relative error in the phase $|S_\varepsilon/S_0 - 1|$ as a function of the frequency mismatch ε can be appreciated in Fig. 6(c). The direct δ -kick strategy yields a relative error that scales as ε^2 , while $\rho \sim \varepsilon^4$ for the spin-echo-like strategy. This latter strategy turns out to be much more robust.

In conclusion, we have developed a general framework to calculate quantitatively a wide variety of phase-space manipulations. Calculations were essentially carried out for a pure state but remain valid for a large class of many-body systems that exhibit self-similar dynamics [27]. We have discussed how a substantial gain in the velocity dispersion reduction can be obtained even when the interaction energy is initially very large in comparison to the kinetic energy. These results are potentially important for the initial-state preparation in many different types of cold-atom experiments.

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