New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

OPEN ACCESS

RECEIVED
29 January 2016

ACCEPTED FOR PUBLICATION
8 March 2016

PUBLISHED
12 April 2016

PAPER

Shortcuts to adiabaticity for an ion in a rotating radially-tight trap

M Palmero[1], Shuo Wang[2], D Guéry-Odelin[3], Jr-Shin Li[2] and J G Muga[1,4]

[1] Departamento de Química Física, UPV/EHU, Apdo. 644, E-48080 Bilbao, Spain
[2] Electrical and Systems Engineering, Washington University in St. Louis, St. Louis, Missouri 63130, USA
[3] Laboratoire de Collisions Agrégats Réactivité, CNRS UMR 5589, IRSAMC, Université de Toulouse (UPS), 118 Route de Narbonne, F-31062 Toulouse CEDEX 4, France
[4] Department of Physics, Shanghai University, 200444 Shanghai, People's Republic of China

E-mail: mikel.palmero@ehu.eus

Keywords: shortcuts to adiabaticity, atom rotation, quantum state engineering

Abstract

We engineer the fast rotation of an effectively one-dimensional ion trap for a predetermined rotation angle and time, avoiding the final excitation of the trapped ion. Different schemes are proposed with different speed limits that depend on the control capabilities. We also make use of trap rotations to create squeezed states without manipulating the trap frequencies.

1. Introduction

A major challenge in modern atomic physics is to develop a scalable architecture for quantum information processing. A proposed scheme to achieve scalability is based on shuttling individual ions or small groups of ions among processing or storing sites [1–6]. Apart from shuttling [7–11] other manipulations of the ion motion would be needed, such as expansions or compressions of the trap [12, 13] separating or merging ion chains [14–16] and rotations [17]. All these basic dynamical operations should fulfill two seemingly contradictory requirements: they should be fast, but free from final motional excitation. Shortcuts to adiabaticity for 'fast and safe driving' have been proposed for several of these elementary operations [7–13, 15] and have also been implemented experimentally [18, 19].

In this paper we study rotations of a single ion as depicted in figure 1. Our aim is to inverse engineer the time-dependence of the control parameter(s) to implement a fast process, free from final excitations. We assume for simplicity that the ion is trapped in a linear, harmonic trap, tightly confined in a radial direction so that it moves effectively along a one-dimensional (1D) axial direction, hereafter 'the line'. The trapping line is set horizontally and is rotated in a time t_f up to an established final angle ($\theta_f = \pi/2$ in all examples) with respect to a vertical axis that crosses the center of the trap. Such an operation would be useful to drive atoms through corners and junctions in a scalable quantum processor [20, 21]. It is also a first step towards the more complicated problem of rotating an ion chain [17, 20, 22], which would facilitate scalability in linear segmented traps, and be useful to rearrange the ions, e.g. to locate a cooling ion at the right position in the chain [17].

We shall first find the classical Hamiltonian. Let s denote a point on the line. s may take positive and negative values. A time dependent trajectory $s(t)$ has Cartesian, laboratory frame components $x = x(s, t)$, $y = y(s, t)$

$$x = s\cos(\theta), \quad y = s\sin(\theta), \tag{1}$$

where $\theta = \theta(t)$ is the rotation angle. The kinetic energy is $K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, where m is the ion mass, and the potential energy is assumed by now to be harmonic, $\frac{1}{2}m\omega_0^2 s^2$ (this will be relaxed below and in section 2), where ω_0 is the angular frequency of the external confining trap in the (longitudinal) direction of the line. This gives the Lagrangian

$$L = \frac{1}{2}m\dot{s}^2 - \frac{1}{2}m\omega^2 s^2, \tag{2}$$

$$\omega^2 = \omega_0^2 - \dot{\theta}^2. \tag{3}$$

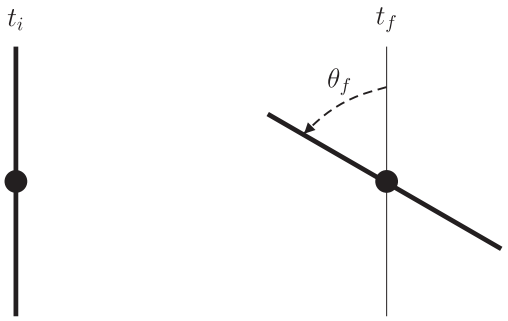

Figure 1. Schematic representation of the rotation process. The ion is confined along a line (where it is subjected to an effective one-dimensional-longitudinal-potential), which is rotated by an angle θ up to θ_f in a time t_f, so that the final state is not excited.

Note that the angular velocity of the rotation $\dot{\theta}$ must be real but could be negative, whereas ω^2 may be positive or negative, making ω purely imaginary in the later case. Unless stated otherwise, the following physically motivated boundary conditions are also assumed: the initial and final trap should be at rest, and we also impose continuity of the angular velocity

$$\theta(0) = 0, \quad \theta(t_f) = \theta_f, \tag{4}$$

$$\dot{\theta}(0) = \dot{\theta}(t_f) = 0, \tag{5}$$

$$\omega(0) = \omega(t_f) = \omega_0, \tag{6}$$

where the last line follows from the second one using equation (3). By a Legendre transformation we finally get the Hamiltonian[5]

$$H = \dot{s}\frac{\partial L}{\partial \dot{s}} - L = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}m\omega^2 s^2. \tag{7}$$

At this point, we quantize this Hamiltonian by substituting $m\dot{s}$ by the momentum operator p and by considering s as the position operator, which becomes a c-number in coordinate representation

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 s^2. \tag{8}$$

We will from now on work with this quantum Hamiltonian (possibly with a more general potential) and corresponding quantum states. It represents formally a harmonic oscillator with time-dependent frequency, but there are significant differences with an actual harmonic oscillator when the inverse engineering of $\omega(t)$ is considered. For an actual harmonic oscillator a fast and safe expansion or compression in a time t_f should take the system from an initial value to a final value of ω without final excitation, in principle without further conditions. By contrast, in the rotation process, according to equation (6), the initial and final effective frequencies are the same, but the conditions in equations (4) and (5) must be satisfied. This implies an integral constraint on ω

$$\theta(t_f) = \int_0^{t_f} \dot{\theta}\mathrm{d}t' = \int_0^{t_f} [\omega_0^2 - \omega^2]^{1/2}\mathrm{d}t', \tag{9}$$

where the square root branch should be chosen to satisfy continuity. One further difference is that in a physical expansion/compression $\omega(t)$ is controlled directly whereas in the rotation there are several options. If ω_0 is constant, only $\dot{\theta}(t)$ is controlled, so that $\omega(t)$ is an 'effective' frequency. In general both ω_0 and $\dot{\theta}$ could be controlled as time-dependent functions, see the next section. As for the final excitation, the expression for the energy of a state that begins in the nth eigenstate of the trap at rest can be found making use of the Lewis–Riesenfeld invariants [12, 23] see the corresponding time-dependent wave function in the appendix

$$\langle H(t)\rangle_n = \frac{(2n+1)\hbar}{4\omega(0)}\left(\dot{b}^2 + \omega^2(t)b^2 + \frac{\omega(0)^2}{b^2}\right). \tag{10}$$

Here b is a scaling factor, proportional to the width of the invariant eigenstates, that satisfies the Ermakov equation

[5] This is easily generalized for a potential $U(s)$, not necessarily harmonic, as $H = \frac{1}{2}m\dot{s}^2 + U(s) - \frac{1}{2}m\dot{\theta}^2 s^2$.

$$\ddot{b} + \omega^2(t)b = \frac{\omega^2(0)}{b^3}. \tag{11}$$

To avoid any final excitation, it is required that

$$b(t_f) = 1, \quad \dot{b}(t_f) = 0 \tag{12}$$

for the initial conditions $b(0) = 1$, $\dot{b}(0) = 0$. The boundary conditions for b and equations (4)–(6) imply that $H(0) = H(t_f)$ commutes with the corresponding Lewis–Riesenfeld invariant [23], so that the nth initial eigenstate is dynamically mapped onto itself (but rotated) at time t_f. In equations (10) and (11) both the excitation energy and the wave packet width are mass independent, so that inverse-engineered rotation protocols will be independent of the species. In the following sections we shall analyze different methods to perform the rotation without final excitation.

2. Control of trap frequency and angular velocity

If both the trap angular frequency ω_0 and the angular velocity $\dot{\theta}$ are controllable functions of time, a simple family of solutions to the inverse problem is found by setting a $\dot{\theta}(t)$ that satisfies equations (4) and (5), and compensating the time dependence of $\dot{\theta}^2$ with a corresponding change in $\omega_0^2(t)$, so that $\omega^2(t) = \omega^2(0)$ remains constant during the whole process. From the point of view of the effective harmonic oscillator 'nothing happens' throughout the rotation, so that the effective state remains unexcited at all times.

We may apply the Lewis–Leach theory of quadratic in momentum invariants [24, 25] to extend the above results to arbitrary potentials[6]. The family of Hamiltonians

$$H = \frac{p^2}{2m} + \frac{1}{2}m\Omega^2 s^2 + \frac{1}{b^2}U\left(\frac{s}{b}\right), \tag{13}$$

where U is an arbitrary function, and Ω depends on time, has the invariant

$$I = \frac{\pi^2}{2m} + \frac{1}{2}m\Omega_0^2 s^2 + U\left(\frac{s}{b}\right), \tag{14}$$

where $\pi = bp - m\dot{b}s$, and Ω_0 is a constant, provided the Ermakov equation

$$\ddot{b} + \Omega^2 b = \frac{\Omega_0^2}{b^3} \tag{15}$$

is satisfied. Consider the simple case $\Omega_0 = 0$, i.e., from equation (15)

$$\Omega^2(t) = -\frac{\ddot{b}}{b}. \tag{16}$$

If we set $b(t) = 1$ as a constant for all times, it follows that $\Omega(t) = 0$. However, as we saw in the previous section, the rotation of a line with the potential $U(s)$ produces in the line frame a centrifugal term $-\dot{\theta}^2 s^2 m/2$. To cancel the total harmonic term, we have to add to the trap potential a compensating harmonic term, $\omega_c^2\, s^2\, m/2$, such that $\omega_c^2 = \dot{\theta}^2$. In other words, $\Omega^2 = \omega_c^2 - \dot{\theta}^2 = 0$. The resulting Hamiltonian and invariant (in this case they are equal) are simply

$$H = I = \frac{p^2}{2m} + U(s), \tag{17}$$

i.e., time independent. No excitation occurs at any time in spite of the fact that a rotation is taking place.

For some applications it may be interesting to consider in equation (13) the more general case in which b depends on time (for example to achieve a squeezed state), and $\omega^2 = \omega_c^2 - \dot{\theta}^2$, corresponding to an auxiliary harmonic term and the centrifugal term. The inverse engineering in this case proceeds by designing $\theta(t)$, so that $\dot{\theta}(0) = \dot{\theta}(t_f) = 0$, and then $b(t)$ obeying the boundary conditions

$$b(0) = b(t_f) = 1, \tag{18}$$

$$\dot{b}(0) = \dot{b}(t_f) = 0, \tag{19}$$

$$\ddot{b}(0) = \ddot{b}(t_f) = 0, \tag{20}$$

(or more generally $b(t_f) = \gamma$) that guarantee the commutation between invariant and Hamiltonian at boundary times. Once θ and b are set we design the auxiliary harmonic term considering, as before, $\Omega_0 = 0$ in equation (15)

[6] The theory was first formulated for classical systems in [24] but is applicable to quantum systems as well [25]. Incidentally this means that the rotation protocols designed in this paper—in this and the following sections—are valid for classical particles as well. The difference appears only when considering which states are valid or not for classical and quantum particles, e.g., when using phase-space formulations of quantum states and classical ensembles.

$$\omega_c^2 = \Omega^2 + \dot{\theta}^2 = -\frac{\ddot{b}}{b} + \dot{\theta}^2. \tag{21}$$

The auxiliary harmonic term vanishes at both boundary times according to the boundary conditions imposed on $\ddot{b}$ and $\dot{\theta}$. In fact Ω^2 vanishes as well at the boundary times so that before and after the rotation the atom is confined only in the potential $U(s)$. This type of protocols, where both the rotation speed and the potential have to be controlled (the latter in space and time) may be quite demanding experimentally. In the rest of the paper we shall assume the simpler scenario in which only the rotation speed $\dot{\theta}$ is controlled, and the trap potential is purely harmonic with constant angular frequency ω_0.

3. Bang–bang

It is possible to perform rotations without final excitation satisfying equations (4) and (5) keeping $\dot{\theta}$ constant or piecewise constant. Here we consider the simplest one-step case

$$\dot{\theta}(t) = \begin{cases} 0, & t \leqslant 0, \\ c, & 0 < t \leqslant t_f, \\ 0, & t \geqslant t_f. \end{cases} \tag{22}$$

Note that equations (5) and (6) are only satisfied now as one-sided limits. A bang–bang approach may admitedly be difficult to implement because of the sharp changes involved, but it sets a useful, simple reference for orders of magnitude estimates of rotation speeds which may be compared to smoother approaches that will be presented later. Integrating $\dot{\theta}$ we find

$$\theta_f = ct_f. \tag{23}$$

For a constant $\dot{\theta} = c$, ω remains constant from $t = 0$ to $t = t_f$, and equal to $\omega_1 = (\omega_0^2 - c^2)^{1/2}$, whereas $\omega = \omega_0$ in the initial and final time regions. For this configuration, and $0 < t < t_f$

$$b(t) = \sqrt{\frac{\omega_0^2 - \omega_1^2}{\omega_1^2}\sin^2(\omega_1 t) + 1}\,, \tag{24}$$

$$\dot{b}(t) = \frac{\sin(\omega_1 t)\cos(\omega_1 t)(\omega_0^2 - \omega_1^2)}{\omega_1 b(t)}, \tag{25}$$

to satisfy the boundary conditions $b(0) = 1$, $\dot{b}(0) = 0$. The shortest final time to satisfy the conditions (12) at t_f is π/ω_1. From equation (23) this gives the value of c needed

$$c = \frac{\theta_f \omega_0}{[\pi^2 + \theta_f^2]^{1/2}}, \tag{26}$$

whereas

$$t_f = \frac{\pi}{\omega_1} = \frac{\pi}{\sqrt{\omega_0^2 - c^2}} = \frac{\pi}{\omega_0} f, \tag{27}$$

$$f := \sqrt{1 + \frac{\theta_f^2}{\pi^2}}. \tag{28}$$

As $c < \omega_0$ the effect of this bang–bang protocol is to expand the effective trap during the rotation time interval. b increases first and then decreases during half an oscillation period of the effective trap. This does not in general coincide with half oscillation period of the actual non-rotating trap π/ω_0 because of the f factor, but it is not too different for relevant values of θ_f. In particular, for $\theta_f = \pi/2, f = 1.118$. The maximum of $b(t)$ at $t_f/2$ is precisely f. For example, for a frequency $\omega_0/(2\pi) = 2$ MHz, this implies a final time $t_f = 0.28\ \mu$s.

4. Optimal control by Pontryagin's maximum principle

While the previous bang–bang method with just one time segment provides a simple guidance, we are also interested in knowing the absolute time minimum that could in principle be achieved (even if the 'optimal' protocol ends up being hardly realizable). Unlike ordinary expansions/compressions, the shortest time protocol for bounded control is not of a bang–bang form. To find it we first rescale the time with ω_0 by setting $\sigma = \omega_0 t$ for $t \in [0,\ t_f]$. Now we set the variables

$$x_1(\sigma) = b(t) = b\left(\frac{\sigma}{\omega_0}\right),$$
$$x_2(\sigma) = \frac{1}{\omega_0}\dot{b}\left(\frac{\sigma}{\omega_0}\right),$$
$$x_3(\sigma) = \int_0^{\sigma} u(\tau)\mathrm{d}\tau, \tag{29}$$

where $u(\sigma) = u(\omega_0 t) = \frac{1}{\omega_0}\dot{\theta}(t)$ with $\sigma \in [0,\ \omega_0 t_\mathrm{f}]$. Then, we can write a control system describing the Ermakov equation (15) and the constraints in (4)–(6), and formulate the time-optimal control (OC) problem for rotation of a quantum particle on a line as

$$\begin{aligned} \min_u J &= \int_0^T 1\mathrm{d}\tau, \\ \text{such that}\quad x_1' &= x_2, \\ x_2' &= \frac{1}{x_1^3} + (u^2 - 1)x_1, \\ x_3' &= u, \end{aligned} \tag{30}$$

where $T = \omega_0 t_\mathrm{f}$ and the prime is a derivative with respect to σ, with the boundary conditions

$$\begin{aligned} x_1(0) &= 1, \quad x_1(T) = 1, \\ x_2(0) &= 0, \quad x_2(T) = 0, \\ x_3(0) &= 0, \quad x_3(T) = \theta_\mathrm{f}. \end{aligned} \tag{31}$$

Note that we assume that the boundary conditions for u at $t = 0$ and $t = t_\mathrm{f}$ can be fulfilled by the use of a sudden switch.

4.1. Unbounded control

We apply the Pontrygin's maximum principle [26] to solve the time-OC problem (30), where the Hamiltonian is given by

$$H(t,\ x,\ u,\ \lambda) = \lambda_0 + \lambda_1 x_2 + \lambda_2\left[\frac{1}{x_1^3} + (u^2 - 1)x_1\right] + \lambda_3 u, \tag{32}$$

in which $\lambda = (\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3)$ and λ_0 is either 0 or 1. The necessary condition $\frac{\partial H}{\partial u} = 0$ gives

$$u^* = -\frac{\lambda_3}{2\lambda_2 x_1}, \tag{33}$$

which minimizes the Hamiltonian and where the co-states $\lambda_1,\ \lambda_2,\ \lambda_3$: $[0,\ T] \to \mathbb{R}$ satisfy $\lambda_i' = -\frac{\partial H}{\partial x_i}$, $i = 1,\ 2,\ 3$, i.e.

$$\begin{aligned} \lambda_1' &= \left[\frac{3}{x_1^4} - (u^2 - 1)\right]\lambda_2, \\ \lambda_2' &= -\lambda_1, \\ \lambda_3' &= 0. \end{aligned} \tag{34}$$

Solutions are found by solving the equation system composed by equations (30), (33) and (34) with the boundary conditions at $\sigma = 0$ in equation (31). We have the freedom of choosing different initial values for the $\lambda_i(0)$ to satisfy the boundary conditions at T in equation (31). We apply a shooting method and numerically minimize $[x_1(T) - 1]^2 + x_2(T)^2 + [x_3(T) - \theta_\mathrm{f}]^2$ for these parameters using MATLAB's 'fminsearch' function with $\theta_\mathrm{f} = \pi/2 = 1.5708$. The best results obtained are for $T = 2.2825$, which, for the external trap frequency $\omega_0/(2\pi) = 2$ MHz used in other examples, implies a final time $t_\mathrm{f} = 0.18\ \mu$s. The solution found is not exact, $(x_1(T),\ x_2(T),\ x_3(T)) = (1.0765,\ 0.0842,\ 1.5650)$, which might be an indication that the system is not controllable. Figure 2 (a) shows the time evolution of u for this case following equation (33) but forcing it to be 0 in the boundary times.

4.2. Bounded control

Now, consider a bounded control with $u(\sigma) \in [0,\ 1]$ for all $\sigma \in [0,\ T]$. Because the Hamiltonian (32) is quadratic in u, the optimal control that minimizes H is of the form

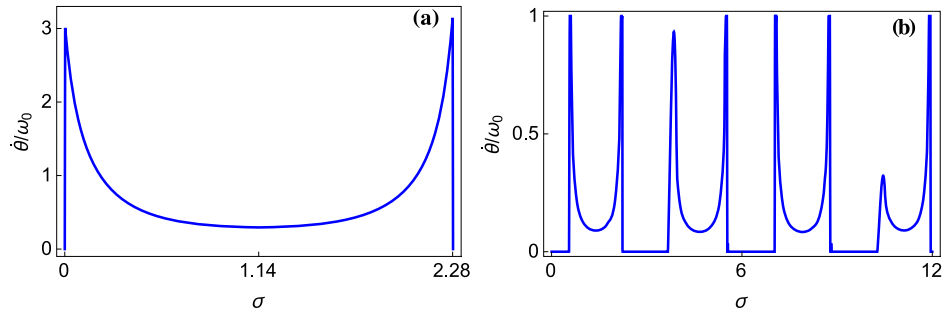

Figure 2. Time evolution $\dot{\theta}(t)$ for the optimal unbounded (a) and bounded (b) control. The rotation angle is $\theta_f = \frac{\pi}{2}$.

$$u_b^* = \min\left\{\max\left\{-\frac{\lambda_3}{2\lambda_2 x_1}, 0\right\}, 1\right\}. \tag{35}$$

The bounded time-optimal control and the resulting optimal trajectory are illustrated in figure 2 (b). The minimum (dimensionless) time that completes the desired rotation is $T = 11.9984$ and the calculated final state following the optimal control is $(x_1(T), x_2(T), x_3(T)) = (1.0083,\ 0.0382,\ 1.5708)$. For $\omega_0/(2\pi) = 2$ MHz, the minimal time is 0.95 μs. Since $u(\sigma) \in [0,\ 1], \forall\, \sigma \in [0,\ T]$, from (30) we see that $\dot{\theta} > 0$, and hence the rotation is always forward. In this case, x_3 reaches the desired $\theta_f = \pi/2$ at $\sigma = 11.9028$, and the control is turned off. Then, the states x_1 and x_2 are oscillating to reach the desired terminal state $(1,\ 0)$. Figure 2 (b) shows the time evolution of u for this solution.

5. Smooth inverse engineering

An alternative inversion route that provides smooth solutions is depicted in the following scheme

$$\theta \longrightarrow \dot{\theta} \longrightarrow \omega \longrightarrow E[b(t_f), \dot{b}(t_f)]$$
$$\text{minimize } E$$

First, $\theta(t)$ is designed to satisfy equations (4) and (5) with some free parameters. The corresponding $\dot{\theta}$ and final energy are calculated, and the parameters are changed until the minimum energy (and excitation) is found.

A convenient choice for θ is a fifth order polynomial ansatz $\theta = \sum_{n=0}^{5} a_n t^n / t_f^n$. In order to satisfy the boundary conditions in equations (4) and (5) we need to fix parameters $a_{0-3} = (0,\ 0,\ a_4 + 2a_5 + 3\theta_f,\ -2a_4 - 3a_5 - 2\theta_f)$. The other two parameters, $a_4,\ a_5$, are left free in order to satisfy the remaining two boundary conditions in equation (12) and suppress the final excitation energy. In practice we solve numerically equation (11) to find the final energy (10) for each pair $a_4,\ a_5$, and use MATLAB's 'fminsearch' function to find the values of the free parameters that minimize the final excitation energy.

In figure 3 the values of the free parameters that result from this process are given, and in figure 4 we depict the corresponding excess energy with respect to the ideal target state (as in previous examples, $\omega_0/(2\pi) = 2$ MHz). Vanishing residual excitations are found for times shorter than half an oscillation period up to a time $t_f \sim 0.23\mu$s, not much larger than the unbounded-optimal-control minimum of 0.18 μs. Figure 5 depicts the difference between the ideal value of $b(t_f)$ and the actual value, and makes evident the sharp change that marks the shortest time for which a solution exists. Since we have limited the possible solutions by imposing a functional form of the function $\theta(t)$, this time is larger than the one found via OC. Note also that the shortest final time is slightly better than the one provided by the simple bang–bang protocol. Table 1 summarizes the results.

6. Wave packet squeezing

Consider now a trap rotation with constant trap frequency ω_0 satisfying the conditions (4)–(6), and b satisfying

$$\begin{aligned} b(0) = 1, \quad \dot{b}(0) = 0, \\ b(t_f) = \gamma, \quad \dot{b}(t_f) = 0. \end{aligned} \tag{36}$$

Unlike the previous sections, b ends in a value γ different from 1.

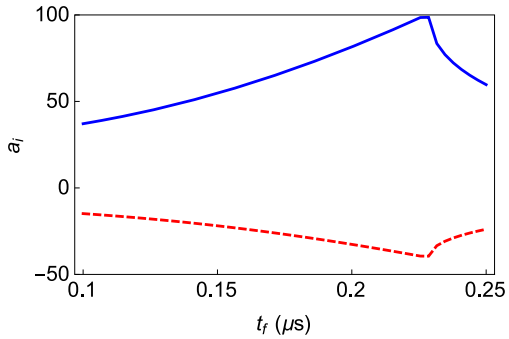

Figure 3. Values of the optimizing parameters a_4 (thick blue line) and a_5 (dashed red line) for different rotation times t_f. The trap frequency is $\omega_0/(2\pi) = 2$ MHz, and the final angle $\theta_f = \frac{\pi}{2}$.

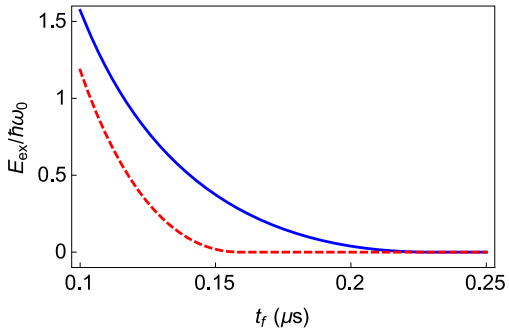

Figure 4. Final excitation energy versus final time for the optimized protocol without (solid blue line) and with final squeezing ($\gamma^2 = 3$, dashed red line). The trap frequency is $\omega_0/(2\pi) = 2$ MHz, and the final rotation angle $\theta_f = \frac{\pi}{2}$. The initial state is the ground state of the trap.

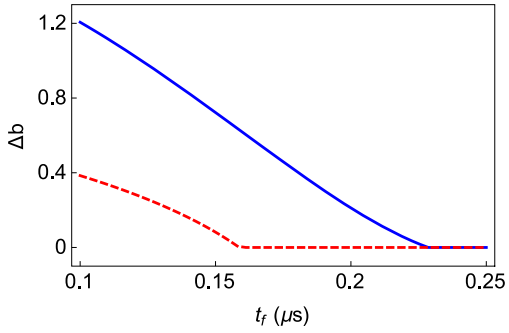

Figure 5. Difference between ideal and actual value of b at the end of the rotation versus final time for the optimized inverse-engineered protocol for rotations without (solid blue line) and with final squeezing ($\gamma^2 = 3$, dashed red line). The trap frequency is $\omega_0/(2\pi) = 2$ MHz, and the final rotation angle $\theta_f = \frac{\pi}{2}$.

Table 1. Minimal rotation times for the different methods. Trap frequency $\omega_0/(2\pi) = 2$ MHz. In bounded OC, $0 \leqslant \dot{\theta} \leqslant \omega_0$.

	Bang–bang	OC(unbounded)	OC(bounded)	Inverse engineering
$t_f(\mu s)$	0.28	0.18	0.95	0.23

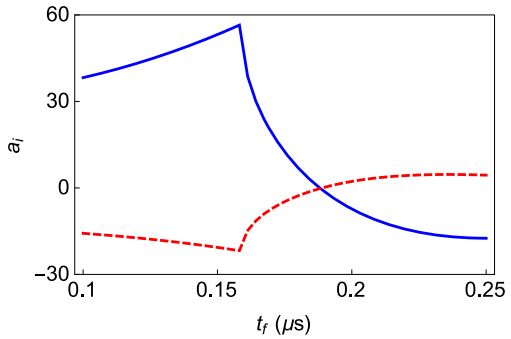

Figure 6. Values of the optimizing parameters a_4 (thick blue line) and a_5 (dashed red line) for different rotation times to generate a squeezed vaccum state with $\gamma^2 = 3$. The trap frequency is $\omega_0/(2\pi) = 2$ MHz, and the rotation angle $\theta_f = \frac{\pi}{2}$.

According to equation (A3), each initial state $\phi_n(0)$ will evolve into $e^{-i(n+1/2)\omega_0 g}\phi_{n,sq}$ at t_f, where $g = g(t_f) = \int_0^{t_f} dt' \big/ b^2(t')$, and $\phi_{n,sq}$ is the normalized eigenstate for the trap with angular frequency $\omega_{sq} = \omega_0/\gamma^2$. (This is a virtual trap, let us recall that the actual trap has angular frequency ω_0.)

A coherent state at time $t = 0$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\phi_n(0)\rangle, \tag{37}$$

will thus evolve into

$$|\psi(t_f)\rangle = e^{-i\omega_0 g/2} e^{-|\tilde{\alpha}|^2/2} \sum_{n=0}^{\infty} \frac{\tilde{\alpha}^n}{\sqrt{n!}} |\phi_{n,sq}\rangle, \tag{38}$$

where $\tilde{\alpha} = \alpha e^{-i\omega_0 g}$. This is a coherent state for the virtual frequency ω_{sq} and therefore a minimum-uncertainty-product state. However, since the actual trap has frequency ω_0, it is also a squeezed coherent state with respect to the actual trap $|[r, \tilde{\alpha}]\rangle$, see [28], where $r = -\ln\gamma$, up to a global phase factor. The final and initial coordinate and momentum widths are related by $\Delta_{s,t_f} = \gamma\Delta_{s,0}$, $\Delta_{p,t_f} = \Delta_{p,0}/\gamma$. We may rewrite the state at time t_f in terms of the squeezing and displacement operators as

$$|\psi(t_f)\rangle = e^{-i\omega_0 g/2} S(r)|\tilde{\alpha}\rangle = e^{-i\omega_0 g/2} S(r) D(\tilde{\alpha})|0\rangle = e^{-i\omega_0 g/2}|[r, \tilde{\alpha}]\rangle, \tag{39}$$

where $S(r) = e^{\frac{r}{2}(a^2 - a^{\dagger 2})}$, a and $a^\dagger$ are annihilation and creator operators for the ω_0-harmonic trap, and $D(z) = e^{za^\dagger - z^* a}$ is the displacement operator. Note that the phase at t_f, $\arg(\tilde{\alpha})$, is controllable by means of the g-function that depends on the process history, whereas the squeezing parameter $1/\gamma$ is controlled by the imposed boundary condition. If necessary, a controlled tilt of the squeezed state in phase space is easy to achieve by letting it evolve, after its formation at t_f, in the fixed, non-rotating trap.

As a simple example let us consider the generation of squeezed vacuum states starting from the ground state of the initial trap, so that $\alpha = 0$. To design the squeezing process we may follow a similar procedure as in the previous section, but minimizing the cost function

$$F = \dot{\tilde{b}}(t_f)^2 + \omega^2(t_f)\tilde{b}(t_f)^2 + \frac{\omega(0)^2}{\tilde{b}(t_f)^2},$$
$$\tilde{b} = b - \gamma + 1, \tag{40}$$

which is minimal for $\tilde{b}(t_f) = 1$ and $\dot{\tilde{b}}(t_f) = 0$, so that $b(t_f) = \gamma$ and $\dot{b}(t_f) = 0$.

Since, due to the centrifugal force during the rotation, the wave packet tends to spread first, the squeezed states with $\gamma > 1$ may be achieved in shorter times than the ones needed without squeezing in the previous section. Figure 6 depicts the free parameters that optimize a rotation with a final squeezed state for the same parameters in the previous subsection, but $\gamma^2 = 3$, and figure 4 the excess energy with respect to the target state. The excitation in a process with a final moderate squeezing is smaller than for the simple rotation without squeezing. Figure 5 depicts the difference between the target value of the function b (proportional to the width of the wavepacket) and its actual value at final time for rotations without and with squeezing. Again, the minimizations change suddenly to a different solution that cannot satisfy the conditions at a critical time, see also figures 3 and 6.

7. Discussion

We have worked out different schemes to perform fast rotations of a 1D trap without any final excitation of the confined particle, which we have considered to be an ion throughout but could be a neutral particle as well by setting the proper trapping interaction. Apart from excitation-free rotations it is also possible to generate squeezed states in a controllable way. For an arbitrary trap, the fast processes could in principle be performed in an arbitrarily short time if an auxiliary harmonic potential with time dependent frequency could be implemented. In a simpler setting, where only the rotation speed may be controlled, the rotation time cannot be arbitrarily short, as demonstrated by inverse engineering or bang–bang approaches, and confirmed by OC theory. Bang–bang and OC protocols provide useful information and time bounds but are difficult to implement experimentally due to the sudden kicks in the angular velocity of the trap. Smooth protocols designed by invariant-based inverse engineering have also been worked out. They achieve negligible excitations for times close to the minimum times given by OC theory.

The analysis may be generalized for a two-dimensional (2D) trap but it becomes considerably more involved [27] and will be considered separately. The 1D approximation used here will be valid for total energies well below the transversal confinement energy $E_\perp = \hbar\omega_\perp$. For the shortest final times considered in our simulations, excitation energies are never larger than $2\hbar\omega_0$ so that $\omega_\perp \gg \omega_0$ would be enough for their validity.

Rotations are elementary manipulations which together with transport, splitting, and expansions, may help to build a scalable quantum information architecture. In particular, they provide a mechanism for connecting sites by changing transport directions in 2D networks. Rotations have been demonstrated experimentally for trapped ions [17] and improving the capability to control the parameters involved is feasible with state-of-the-art trapped-ion technology. To extend the present analysis to ion chains [17], an approach similar to that in [9, 13, 15] could be applied, working out the dynamical modes of the system and taking into account the dipole–dipole interaction due to the rotation of the charged particles. The present results set a first step towards accurately controlling rotating ion chains which would allow for fast reordering.

Acknowledgments

We thank Kazutaka Takahashi, Joseba Alonso, Uli Poschinger, and Christian Schmiegelow for useful discussions. This work was supported by the Basque Country Government (Grant No. IT472-10), Ministerio de Economía y Competitividad (Grant No. FIS2012-36673-C03-01), the program UFI 11/55 of the Basque Country University and the US National Science Foundation under the awards CMMI-1301148 and CMMI-1462796. MP acknowledges a fellowship by UPV/EHU.

Appendix. Wave functions

The time-dependent wave functions evolving with the Hamiltonian (8) take the form [7, 12, 23]

$$\langle s|\psi(t)\rangle = \sum_n c_n e^{i\alpha_n(t)}\langle s|\phi_n(t)\rangle, \tag{A1}$$

where the c_n are constant

$$\alpha_n(t) = -\frac{1}{\hbar}\int_0^t dt' \frac{(n+1/2)\hbar\omega_0}{b^2} = -\omega_0(n+1/2)\int_0^t dt' \frac{1}{b^2}, \tag{A2}$$

$$\langle s|\phi_n(t)\rangle = e^{\frac{im}{\hbar}\dot{b}q^2/(2b)}\frac{1}{b^{1/2}}\Phi_n(s/b) \tag{A3}$$

and $\Phi_n(x)$ is the Hermite polynomial solution of the harmonic oscillator with angular frequency ω_0 and energy eigenvalue $(n+1/2)\hbar\omega_0$, $\Phi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{\frac{-m\omega_0 x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right)$. Note that $\frac{1}{b^{1/2}}\Phi_n(s/b)$ is just a scaled state which corresponds to the nth eigenstate of a trap with angular frequency ω_0/b^2.

References

[1] Kielpinski D, Monroe C and Wineland D J 2002 Architecture for a large-scale ion-trap quantum computer *Nature* **417** 709
[2] Rowe M A *et al* 2002 Transport of quantum states and separation of ions in a dual rf ion trap *Quant. Inf. Comput.* **2** 257
[3] Reichle R, Leibfried D, Blakestad R B, Britton J, Jost J D, Knill E, Langer C, Ozeri R, Seidelin S and Wineland D J 2006 Transport dynamics of single ions in segmented microstructured Paul trap arrays *Fortschr. Phys.* **54** 666
[4] Home J P, Hanneke D, Jost J D, Amini J M, Leibfried D and Wineland D J 2009 Complete methods set for scalable ion trap quantum information processing *Science* **325** 1227
[5] Roos C 2012 Moving traps offer fast delivery of cold ions *Physics* **5** 94
[6] Monroe C and Kim J 2013 Scaling the ion trap quantum processor *Science* **339** 1164

[7] Torrontegui E, Ibáñez S, Xi Chen, Ruschhaupt A, Guéry-Odelin D and Muga J G 2011 Fast atomic transport without vibrational heating *Phys. Rev.* A **83** 013415
[8] Palmero M, Torrontegui E, Guéry-Odelin D and Muga J G 2013 Fast transport of two ions in an anharmonic trap *Phys. Rev.* A **88** 053423
[9] Palmero M, Bowler R, Gaebler J P, Leibfried D and Muga J G 2014 Fast transport of mixed-species ion chains within a Paul trap *Phys. Rev.* A **90** 053408
[10] Lu X-J, Muga J G, Xi Chen, Poschinger U G, Scmidt-Kaler F and Ruschhaupt A 2014 Fast shuttling of a trapped ion in the presence of noise *Phys. Rev.* A **89** 063414
[11] Lu X-J, Palmero M, Ruschhaupt A, Xi Chen and Muga J G 2015 Optimal transport of two ions under slow spring-constant drifts *Phys. Scripta* **90** 074038
[12] Chen Xi Ruschhaupt A, Schmidt S, del Campo A, Guéry-Odelin D and Muga J G 2010 Fast optimal frictionless atom cooling in harmonic traps: shortcut to adiabaticity *Phys. Rev. Lett.* **104** 063002
[13] Palmero M, Martínez-Garaot S, Alonso J, Home J P and Muga J G 2015 Fast expansions and compressions of trapped-ion chains *Phys. Rev.* A **91** 053411
[14] Lau H-K and James D F V 2012 Proposal for a scalable universal Bosonic simulator using individually trapped ions *Phys. Rev* A **85** 062329
[15] Palmero M, Martínez-Garaot S, Poschinger U, Ruschhaupt A and Muga J G 2015 Fast separation of two trapped ions *New J. Phys.* **17** 093031
[16] Kaufmann H, Ruster T, Schmiegelow C T, Schmidt-Kaler F and Poschinger U G 2014 Dynamics and control of fast ion crystal splitting in segmented Paul traps *New J. Phys.* **16** 073012
[17] Splatt F, Harlander M, Brownutt M, Zähringer F, Blatt R and Hänsel W 2009 Deterministic reordering of $^{40}Ca^{+}$ ions in a linear segmented Paul trap *New J. Phys.* **11** 103008
[18] Bowler R, Gaebler J P, Lin Y, Tan T R, Hanneke D, Jost J D, Home J P, Leibfried D and Wineland D J 2012 Coherent diabatic ion transport and separation in a multizone trap array *Phys. Rev. Lett.* **109** 080502
[19] Walther A, Ziesel F, Ruster T, Dawkins S T, Ott K, Hettrich M, Singer K, Schmidt-Kaler F and Poschinger U 2012 Controlling fast transport of cold trapped ions *Phys. Rev. Lett.* **109** 080501
[20] Hensinger W K *et al* 2006 T-junction ion trap array for two-dimensional ion shuttling, storage and manipulation *Appl. Phys. Lett.* **88** 034101
[21] Amini J M, Uys H, Wesenberg J H, Seidelin S, Britton J, Bollinger J J, Leibfried D, Ospelkaus C, VanDevender A P and Wineland D J 2010 Toward scalable ion traps for quantum information processing *New J. Phys.* **12** 033031
[22] Blakestad R B, VanDevender A P, Ospelkaus C, Amini J M, Britton J, Leibfried D and Wineland D J 2009 High fidelity transport of trapped-ion qubits through an X-junction trap array *Phys. Rev. Lett.* **102** 153002
[23] Lewis H R and Riesenfeld W B 1969 An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field *J. Math. Phys.* **10** 1458
[24] Lewis H R and Leach P G 1982 A direct approach to finding exact invariants for one-dimensional time-dependent classical Hamiltonians *J. Math. Phys.* **23** 2371
[25] Dhara A K and Lawande S W 1984 *J. Phys. A: Math. Gen.* **17** 2423
[26] Pontryagin L S 1962 *The Mathematical Theory of Optimal Processes* (New York: Interscience)
[27] Masuda S and Rice S A 2015 Rotation of the orientation of the wave function distribution of a charge particle and its utilization *J. Phys. Chem.* B **119** 11079
[28] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)