Scissors Mode and Superfluidity of a Trapped Bose-Einstein Condensed Gas

D. Guéry-Odelin and S. Stringari
Dipartimento di Fisica, Università di Trento, and Istituto Nazionale per la Fisica della Materia, I-38050 Povo, Italy
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We investigate the oscillation of a dilute atomic gas generated by a sudden rotation of the confining trap (scissors mode). This oscillation reveals the effects of superfluidity exhibited by a Bose-Einstein condensate. The scissors mode is also investigated in a classical gas above T_c in various collisional regimes. The crucial difference with respect to the superfluid case arises from the occurrence of low frequency components, which are responsible for the rigid value of the moment of inertia. Different experimental procedures to excite the scissors mode are discussed.

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Superfluidity is one of the most spectacular consequences of Bose-Einstein condensation and has been the object of extensive experimental and theoretical work in the past, especially in connection with the physics of liquid helium [1]. Indirect signatures of superfluidity in trapped Bose-Einstein condensed gases are given by their dynamic behavior at very low temperatures [2] which well confirms the predictions of the hydrodynamic theory of superfluids, as well as by the occurrence of spectacular interference phenomena [3] which point out the importance of coherence effects, typical of superfluids. However, direct evidence of superfluidity is still missing in these systems.

Important manifestations of superfluidity are associated with rotational phenomena. These include the strong reduction of the moment of inertia with respect to the classical rigid value and the occurrence of quantized vortices [4]. These peculiar features are the direct consequence of the irrotational nature of the superfluid flow and have already been the object of theoretical work also in the case of dilute trapped gases (see, for example, Ref. [5] and references therein).

The purpose of this paper is to focus on the oscillatory behavior exhibited by the rotation of the atomic cloud with respect to the symmetry axis of the confining trap (see Fig. 1) and on the corresponding superfluid effects caused by Bose-Einstein condensation. A similar mode, called the scissors mode, is well known in nuclear physics [6], where it corresponds to the out-of-phase rotation of the neutron and proton clouds, and its recent systematic experimental investigation [7] has confirmed the occurrence of superfluidity in an important class of deformed nuclei.

In the presence of a deformed external potential the restoring force associated with the rotation of the cloud in the x-y plane is proportional to the square of the deformation parameter ϵ of the trap [see Eq. (1) below]. The mass parameter is instead fixed by the moment of inertia. For a superfluid system this is given by the irrotational value and is hence proportional to ϵ^2. As a consequence, the frequency of the oscillation approaches a finite value when the deformation tends to zero. Vice versa, in the absence of superfluidity, the moment of inertia takes the rigid value and the scissors mode exhibits low frequency components.

Let us start our investigation by considering a Bose-Einstein condensed gas at equilibrium in a deformed potential of the form

$$V_{\mathrm{ext}}(\mathbf{r}) = \frac{m}{2}\,\omega_x^2 x^2 + \frac{m}{2}\,\omega_y^2 y^2 + \frac{m}{2}\,\omega_z^2 z^2 \qquad (1)$$

with $\omega_x^2 = \omega_0^2(1+\epsilon)$ and $\omega_y^2 = \omega_0^2(1-\epsilon)$, where ϵ gives the deformation of the trap in the x-y plane. For large enough samples, one can safely use the Thomas-Fermi approximation for the ground state density,

$$n_0(\mathbf{r}) = [\mu - V_{\mathrm{ext}}(\mathbf{r})]/g\,, \qquad (2)$$

where the strength parameter g is related to the scattering length a by the relation $g = 4\pi\hbar^2 a/m$. We consider a gas initially at equilibrium in the (x', y', z) frame. At $t=0$, one rotates abruptly the eigenaxis of the trap from its initial position to (x, y, z) by a small angle $-\theta_0$ (see Fig. 1). As a consequence of the sudden rotation, the system will no longer be in equilibrium and will start oscillating. If the angle θ_0 is not too large the oscillation

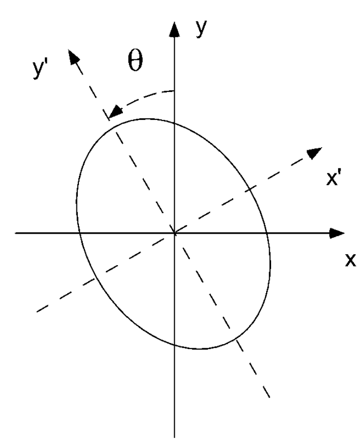

FIG. 1. Exciting the scissor mode: initially the gas is thermalized in an anisotropic trap. One then abruptly rotates the eigenaxis of the trap by a small angle.

will correspond to a rotation of the cloud in the x-y plane (scissors mode). The equations describing this rotation can be easily obtained at zero temperature starting from the time-dependent Gross-Pitaevskii equation for the order parameter which, in the Thomas-Fermi regime, takes the typical form of the hydrodynamic equations of superfluids [8]:

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\mathbf{v}) = 0\,, \tag{3}$$

$$m\frac{\partial \mathbf{v}}{\partial t} + \nabla_{\mathbf{r}}\left(V_{\mathrm{ext}}(\mathbf{r}) + gn + \frac{mv^2}{2}\right) = 0\,. \tag{4}$$

Starting from Eqs. (3) and (4) one easily obtains, in the linear regime, the closed set of equations,

$$\frac{d}{dt}\langle xy\rangle = \langle xv_y + yv_x\rangle\,, \tag{5}$$

$$\frac{d}{dt}\langle xv_y + yv_x\rangle = -2\omega_0^2\langle xy\rangle\,, \tag{6}$$

involving the relevant quadrupole variable $\langle xy\rangle = \int d\mathbf{r}\, xyn(\mathbf{r},t)/N$. The corresponding density profile can be parametrized in the form

$$n(\mathbf{r},t) = [\mu - V_{\mathrm{ext}}(\mathbf{r}) - m\omega_0^2\alpha(t)xy]/g\,, \tag{7}$$

where α is a small time-dependent coefficient related to $\langle xy\rangle$ by

$$\langle xy\rangle = -\frac{2\omega_0^2\alpha(t)}{3m\omega_x^2\omega_y^2}\langle V_{\mathrm{ext}}\rangle_0\,, \tag{8}$$

and$\langle V_{\mathrm{ext}}\rangle_0$ is the expectation value of the harmonic potential at equilibrium. The associated velocity field is irrotational and has the form $\mathbf{v}(\mathbf{r},t) \propto \nabla(xy)$. It is worth noticing that the parametrization (7) will describe a rotation of the sample with the angle fixed by the relation $\alpha(t) = 2\epsilon\theta(t)$ only if $\alpha \ll \epsilon$. If instead α is larger than ϵ, then (7) corresponds to a traditional quadrupole deformation characterized by a change of the intrinsic shape and the connection with the geometry of the scissors is lost. In order to obtain a visible signal of rotational type in the density (7) the deformation parameter should not be consequently too small. Assuming $\alpha \ll \epsilon$ (or, equivalently, $\theta \ll 1$), one finds, from (5), (6), and (8), the equation

$$\frac{d^2\theta(t)}{dt^2} + 2\omega_0^2\theta = 0\,. \tag{9}$$

The solution, corresponding to the chosen initial conditions $\theta(0) = \theta_0$ and $\theta'(0) = 0$, is $\theta = \theta_0\cos(\omega t)$ and oscillates with frequency $\omega = \sqrt{2}\,\omega_0$.

Let us now consider the behavior of the system in the absence of Bose-Einstein condensation. The simplest case is the high-T, classical regime where analytic solutions are available in the framework of the Boltzmann kinetic equations. We will use the method of the averages, already employed to discuss the damping of the quadrupole oscillation [9]. The method provides the following closed set of equations:

$$\frac{d}{dt}\langle xy\rangle = \langle xv_y + yv_x\rangle\,, \tag{10}$$

$$\frac{d}{dt}\langle xv_y - yv_x\rangle = 2\epsilon\omega_0^2\langle xy\rangle\,, \tag{11}$$

$$\frac{d}{dt}\langle xv_y + yv_x\rangle = 2(\langle v_xv_y\rangle - \omega_0^2\langle xy\rangle)\,, \tag{12}$$

$$\frac{d}{dt}\langle v_xv_y\rangle = -\omega_0^2\langle xv_y + yv_x\rangle - \epsilon\omega_0^2\langle xv_y - yv_x\rangle - \frac{\langle v_xv_y\rangle}{\tau}\,, \tag{13}$$

where the averages are taken here in both coordinate and velocity space and the collisional term has been evaluated in the linear regime using a Gaussian approximation for the distribution function [9]. The relaxation time entering (13) is fixed by the relation $\tau = 5/(4\gamma_{\mathrm{coll}})$, where $\gamma_{\mathrm{coll}} = n(0)v_{\mathrm{th}}\sigma/2$ is the classical collisional rate, $n(0)$ is the central density of the atomic cloud, $v_{\mathrm{th}} = \sqrt{8k_BT/\pi m}$ is the thermal velocity, and $\sigma = 8\pi a^2$ is the elastic cross section. Notice that collisions affect only the equation for the variable $\langle v_xv_y\rangle$. In fact the other variables are conserved by the elastic collisions [10]. It is also worth noticing that the angular momentum $m(xv_y - yv_x)$ is not a constant of motion, due to the absence of symmetry in the confining potential and is coupled, through Eq. (11), to the quadrupole variable $\langle xy\rangle$. It is finally interesting to note that Eqs. (10)–(13), with the collisional term set equal to zero, exactly hold also in the case of a noninteracting Bose or Fermi gas. In this case the effects of quantum statistics enter only through the initial conditions of the corresponding dynamic variables. For oscillations of small amplitude the density profile of the classical gas corresponds to the Gaussian parametrization

$$n(\mathbf{r},t) \propto e^{-[V_{\mathrm{ext}}(\mathbf{r}) + \alpha(t)m\omega_0^2xy]/kT}\,, \tag{14}$$

and the expectation value of $\langle xy\rangle$ is given by the same relation (8) previously derived for the $T = 0$ Bose-Einstein condensed gas. The initial conditions corresponding to the sudden rotation of the gas are given by $\langle xy\rangle_{t=0} = \epsilon\theta_0\langle x^2 + y^2\rangle_{t=0}$, $\langle xv_y \pm yv_y\rangle_{t=0} = 0$, and $\langle v_xv_y\rangle_{t=0} = 0$. The vanishing of $\langle xv_y \pm yv_y\rangle_{t=0}$ follows from the absence of currents at $t = 0$, while the vanishing of $\langle v_xv_y\rangle_{t=0}$ is the consequence of the initial isotropy of the velocity distribution. By using the relationship $\alpha(t) = -2\epsilon\theta(t)$ holding for small rotational angles, the equations of motion [(10) and (11)] can be usefully rewritten in the form

$$\left(\frac{d^4\theta}{dt^4} + 4\omega_0^2\frac{d^2\theta}{dt^2} + 4\epsilon^2\omega_0^4\theta\right) + \frac{1}{\tau}\left(\frac{d^3\theta}{dt^3} + 2\omega_0^2\frac{d\theta}{dt}\right) = 0\,, \tag{15}$$

and the initial conditions take the form $\theta(0) = \theta_0$, $\theta''(0) = -2\omega_0^2\theta(0)$, and $\theta'(0) = \theta'''(0) = 0$. Notice that Eq. (15) differs from the corresponding Eq. (9) for the superfluid regime. It gives rise to different solutions propagating at high and low frequency. For small values of the deformation parameter ϵ, the former can be identified with the $\ell_z = 2$ irrotational quadrupole oscillation, i.e., the classical counterpart of the superfluid oscillation described by (9). The latter instead corresponds to the rotational mode of the system and is absent in the superfluid case. Actually, if ϵ becomes too small, the variable θ loses its geometrical meaning. In this case the physical variables are, on the one hand, the moment $\langle xy \rangle$, which characterizes the high-lying quadrupole mode and is coupled with the variables $\langle xv_y + yv_y \rangle$ and $\langle v_x v_y \rangle$, and, on the other hand, the angular momentum $m\langle xv_y - yv_x \rangle$, which becomes a constant of motion [see Eq. (11)].

Let us discuss the different collisional regimes predicted by (15). In the collisionless regime [first term in (15)] the two frequencies are undamped and given, respectively, by $|\omega_x \pm \omega_y|$. In the opposite, hydrodynamic, limit (second term) only the high-lying oscillation survives with frequency $\omega = \sqrt{2}\,\omega_0$, while the low-lying solution becomes overdamped as $\tau \to 0$, in agreement with the diffusive nature of the transverse waves predicted by classical hydrodynamics [11]. A maximum damping is obtained for $\omega_0 \tau \sim 1$, a condition easily achievable in current experiments.

The chosen initial condition, corresponding to a sudden rotation of the sample, gives rise to the excitation of both the low and high frequency modes. The resulting time evolution of the observable $\theta(t)$ is shown in Figs. 2 and 3 for two different collisional regimes and the choice $\epsilon = 0.3$. In Fig. 2 we have made the "collisionless" choice $\epsilon\omega_0\tau \sim 6$, so that both the high-lying and low-lying modes have small damping. The figure clearly shows the combined signals propagating with frequencies $|\omega_x \pm \omega_y|$ respectively. In Fig. 3 we have instead made the choice $\epsilon\omega_0\tau \sim 0.15$ corresponding to large damping. With such a choice the low frequency rotational mode is overdamped and the remaining oscillation exhibited by the curve is due to the high frequency component. The achievement of the overdamped, "hydrodynamic" regime for the low frequency mode is favored by the choice of small values of ϵ.

From the comparison between the curves of Figs. 2 and 3 it emerges very clearly that the main feature of the superfluid regime (dashed lines in Figs. 2 and 3) is the absence of low frequency components. This behavior should be easily verifiable experimentally. It might also provide a useful test of superfluidity in Fermi trapped gases, below the BCS transition.

Until now, we have discussed the superfluid case ($T = 0$) and the classical regime $T > T_c$. Below T_c the system can be described in terms of a two-fluid model, and one needs two different angles to describe the motion of the system. Neglecting interaction effects between the two fluids, the superfluid component would be governed by (9) while the thermal part would evolve according to (15). Of course, in this case the relaxation time τ should be evaluated by taking into account Bose statistics. Inclusion of interaction effects between the condensate and the thermal component would lead to a damping of the superfluid oscillation, similar to what happens in the case of the quadrupole mode [12].

The sudden rotation of the trap is not the only way to excite the scissors mode. In the last part of this Letter we discuss an alternative procedure which further emphasizes the superfluid nature of the Bose-Einstein condensed gas. We consider a gas initially in equilibrium within a trap rotating with frequency Ω. Experimental techniques to achieve rotating configurations of this type are in progress [13]. At the time $t = 0$ we suddenly stop the rotation of the trap, and the gas, due to its inertia, will start rotating. In the absence of superfluidity the moment of inertia is large and the oscillations will be characterized by low frequency components and large amplitudes. Vice versa, if the system is superfluid the moment of inertia is small (proportional to ϵ^2) and the oscillations will be

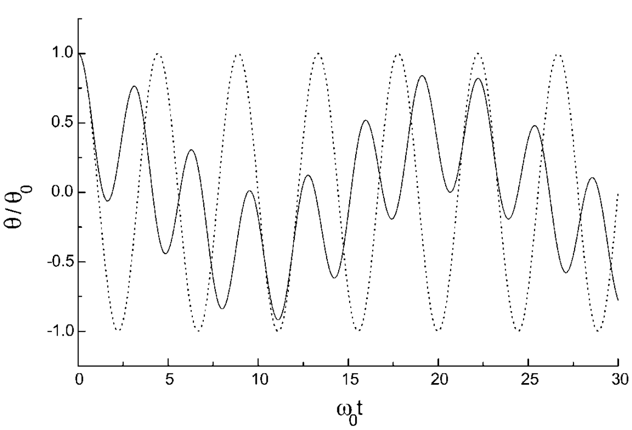

FIG. 2. Angle θ as a function of time for a classical gas (solid line) in the collisionless regime ($\epsilon\omega_0\tau = 6$ and $\epsilon = 0.3$) and for a superfluid regime (dashed line).

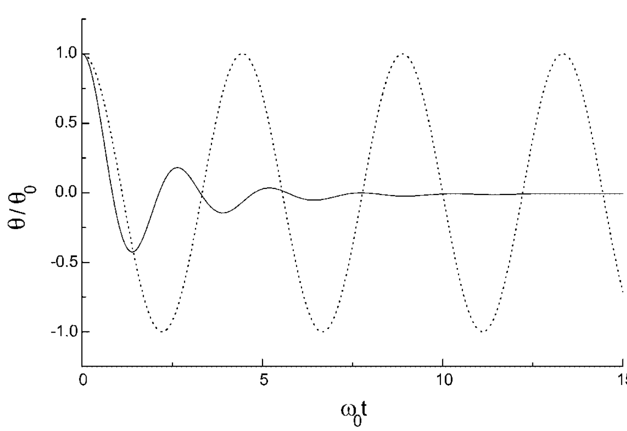

FIG. 3. Angle θ as a function of time for a classical gas (solid line) in the hydrodynamic regime ($\epsilon\omega_0\tau = 0.15$ and $\epsilon = 0.3$) and for a superfluid regime (dashed line).

characterized by large frequencies and small amplitudes. The equations of motion describing the rotation are still given by (5) and (6) but the initial conditions are different, corresponding to the presence of a current term in the system at $t = 0$. One finds $\theta(0) = 0$ and $\theta'(0) = \Omega$. In the classical case, one has the additional conditions $\theta''(0) = \theta'''(0) = 0$. The amplitude of the oscillations, in the collisionless regime, scale as Ω/ω_0 for the superfluid and as $\Omega/(\omega_0\epsilon)$ for the classical gas. The difference is due to the fact that at $t = 0$ the velocity field generated by the rotation is very different in the two cases. In the superfluid case it is given by the irrotational form $\mathbf{v} = -\Omega\epsilon\nabla(xy)$, while in the classical case it is given by the rotational form $\mathbf{v} = \Omega \times \mathbf{r}$. If the amplitude of the oscillation becomes too large the dynamic variable $\langle xy\rangle$ can no longer be simply connected with the angle θ. This easily happens in the classical case, where the amplitude of the oscillation is magnified by the factor $1/\epsilon$.

The excitation of the scissors mode, based on a sudden switching off of the rotation of the trap, would allow also for an explicit determination of the moment of inertia Θ of the gas defined by the linear relation $m\langle xv_y - yv_x\rangle_{t=0} = \Omega\Theta/N$. In fact the knowledge of the time evolution of $\langle xy\rangle$ permits one to calculate directly the value of the angular momentum at $t = 0$. From Eq. (11), which holds for classical as well as for quantum systems, one has $m\langle xv_y - yv_x\rangle_{t=0} = -2\epsilon\omega_0^2 \int d\omega\, F(\omega)/\omega$ with $F(\omega)$ defined by

$$\langle xy\rangle(t) = \int d\omega\, F(\omega)(e^{i\omega t} - e^{-i\omega t})/2i\,. \tag{16}$$

On the other hand, from the model-independent equation (10), one easily finds the result $\langle xv_y + yv_x\rangle_{t=0} = \int d\omega\, F(\omega)\omega$ so that, by using the relation $\langle xv_y + yv_x\rangle_{t=0} = \Omega\langle x^2 - y^2\rangle_{t=0}$ predicted by linear response theory [14], one finally obtains the useful expression,

$$\Theta = \Theta_{\mathrm{rig}}(\omega_y^2 - \omega_x^2)\frac{\langle x^2 - y^2\rangle_{t=0}}{\langle x^2 + y^2\rangle_{t=0}}\frac{\int d\omega\, F(\omega)/\omega}{\int d\omega\, F(\omega)\omega}\,, \tag{17}$$

relating the moment of inertia of the system to the measurable Fourier signal $F(\omega)$. In this equation, $\Theta_{\mathrm{rig}} = Nm\langle x^2 + y^2\rangle$ is the rigid value of the moment of inertia. Notice that Eq. (17) provides an exact relationship for the moment of inertia holding for classical as well as Bose or Fermi interacting systems confined by a harmonic trap of the type (1). If the system oscillates with the single frequency $\omega = \sqrt{2}\,\omega_0$, as happens in the $T = 0$ Thomas-Fermi superfluid regime, one immediately finds the result $\Theta = \epsilon^2\Theta_{\mathrm{rig}}$, corresponding to the irrotational value of the moment of inertia. Vice versa, in a classical gas or in a normal Fermi gas trapped by a harmonic potential, the response is dominated by low frequency components, of order $\epsilon\omega_0$, and Eq. (17) yields the rigid value for the moment of inertia, confirming the results for Θ previously obtained [5] using sum rule techniques.

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Note added.—After submission of this Letter, experimental evidence for superfluidity was reported by Matthews *et al.* [15] and by Raman *et al.* [16]. The scissor mode was observed by Marago *et al.* [17].

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[14] According to linear response theory, the constraint $-\Omega\sum_i \ell_z^{(i)}$ yields the result $\langle xv_y + yv_x\rangle = \Omega\Xi$, where $\Xi = \sum_{nq}(\Pi_n - \Pi_q)\langle n|\sum_i(x^{(i)}v_y^{(i)} + y^{(i)}v_x^{(i)}) \times |q\rangle\langle q|\sum_i \ell_z^{(i)}|n\rangle/(E_q - E_n)$ is the static response function relative to the corresponding operators with $\Pi_n = e^{-E_n/k_BT}/\sum_n e^{-E_n/k_BT}$. Using the model-independent commutation relations $[H,\sum_i x^{(i)}y^{(i)}] = -i\hbar\sum_i(x^{(i)}v_y^{(i)} + y^{(i)}v_x^{(i)})$ and $[\ell_z, xy] = im\hbar(x^2 - y^2)$ as well as the completeness relation, one immediately derives the relationship $\langle xv_y + yv_x\rangle = \Omega\langle x^2 - y^2\rangle$.
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