

## Thermalization in mixtures of ultracold gases

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(Received 28 July 2005; published 3 March 2006)

Starting from a set of coupled Boltzmann equations, we investigate the thermalization of a two-species cold atomic gas confined either in a box or in an isotropic harmonic trap. We show that the thermalization times, by contrast to the collision rate, depend on the interferences between scattering partial waves. The dynamics of thermalization in a harmonic trap is also strongly dependent upon the ratio between the collision rate and the trap frequencies.

DOI: [10.1103/PhysRevA.73.032706](https://doi.org/10.1103/PhysRevA.73.032706)

PACS number(s): 34.10.+x, 05.30.Jp

### I. INTRODUCTION

Ultracold mixtures of alkali-metal atoms have attracted great interest among atomic physicists. The ability to simultaneously cool and trap multiple species offers the possibility to study a large variety of physical phenomena such as, for instance, Fermi-Bose mixtures [Ref. 1].

Thermalization plays a crucial role in the cooling of a sample by evaporation, and/or in the implementation of sympathetic cooling. The detailed knowledge of the underlying physics is all the more important for experiments where the time available to perform evaporation is limited [2], or for the optimization of the evaporation ramp in the multipartial-wave regime [3]. In addition the return to equilibrium of a gas initially prepared in an out-of-equilibrium state is a widely used way to infer the value of the scattering length [4–10].

The purpose of this paper is to provide a theoretical framework to describe the thermalization between two cold gases confined in a box or a harmonic trap. In contrast to previous theoretical studies on this subject [6,7,11], we explicitly take into account all partial waves involved in the collision process, and we work out the role played by the confinement in all collisional regimes. Since the basic mechanism described by the Boltzmann equation is binary collisions, two-component systems exhibit all the complexities of  $n$ -component systems. The generalization of the results we obtain to systems containing more than two species is therefore straightforward.

The paper is organized as follows. Section II describes the theoretical framework used to investigate the dynamics of thermalization. Section III is devoted to the simple case of a gas confined in a box. It gives the explicit expression for the rethermalization time including the interference terms between all partial waves involved in the collision process. Section IV addresses the same problem for a gas trapped in an isotropic harmonic trap.

### II. THEORETICAL FRAMEWORK

The gas  $i$  ( $i=1,2$ ) consists of  $N_i$  particles of mass  $m_i$  initially thermalized at  $T_i^0$ . The dynamics of the gas is de-

scribed by a set of two coupled Boltzmann equations for each phase-space distribution function  $f_i(\mathbf{r}, \mathbf{v}_i, t)$ :

$$\frac{\partial f_i}{\partial t} + \{f_i, H_i\} = I_{ii}[f_i] + I_{ij}[f_i, f_j], \quad i \neq j. \quad (1)$$

The distribution function  $f_i$  for the species  $i$  is governed by the Hamiltonian  $H_i$  through the Poisson bracket  $\{f_i, H_i\}$ , and by binary collisions. The appearance of two collision terms on the right-hand side of Eq. (1) is a result of the possibility of  $f_i$  to change through both self-collisions ( $i$ - $i$  collisions) or cross collisions (1-2). The expression for the collision integrals is

$$I_{ij}[f_i, f_j] = \int d^2\Omega' d^3v_B |\mathbf{v}_B - \mathbf{v}_A| \frac{d\sigma_{ij}}{d^2\Omega'} \\ \times [f_i(\mathbf{r}, \mathbf{v}'_A, t) f_j(\mathbf{r}, \mathbf{v}'_B, t) - f_i(\mathbf{r}, \mathbf{v}_A, t) f_j(\mathbf{r}, \mathbf{v}_B, t)], \quad (2)$$

where  $i, j=1,2$ . They account for elastic collisions between particles labeled  $A$  and  $B$ . We denote the velocities of the two atoms before they interact with each other by  $\mathbf{v}_A$  and  $\mathbf{v}_B$ , and after the collision by  $\mathbf{v}'_A$  and  $\mathbf{v}'_B$ . The expression for the collision terms can be easily extended to include the effects of both Bose and Fermi statistics leading to the Boltzmann-Nordheim equation [12,13]. In fact, most of the results discussed in this paper also hold in the presence of quantum degeneracy, provided the system is not Bose-Einstein condensed. The quantitative estimates of collision effects presented in this work are, however, based on classical statistics. As shown in Ref. [11] for bosons and Ref. [14] for fermions, our approach, based on the result of the classical physics, is valid as soon as the temperature of the sample is larger than twice the temperature for degeneracy, i.e., the critical temperature for bosons and the Fermi temperature for fermions.

The collision problem is simplified by an appropriate change of variables. We introduce the center-of-mass velocity  $\mathbf{v}_0 = (m_A \mathbf{v}_A + m_B \mathbf{v}_B) / (m_A + m_B)$ , the total mass  $M = m_A + m_B$ , the relative velocity  $\mathbf{v}_r = \mathbf{v}_A - \mathbf{v}_B$ , and the reduced mass  $\mu = m_A m_B / M$ . The relative velocity changes from the value  $\mathbf{v}_r$

before the collision to the value  $\mathbf{v}'_r$  after the collision, keeping the same value for the modulus  $v_r=v'_r$  because of energy conservation. The collision process is described by specifying the scattering direction with the polar angle  $\theta'$  and the azimuthal angle  $\varphi'$  of the final relative velocity  $\mathbf{v}'_r$  with respect to the initial relative velocity  $\mathbf{v}_r$  before the collision. The corresponding solid angle  $d^2\Omega' = d\varphi' \sin\theta' d\theta'$  enters the expression for the integrand of the collision integrals.

The angular dependence of the elastic scattering directly derives from the partial-waves decomposition of the scattering amplitude in the quantum-mechanical description of interactions [15]. For nonidentical atoms, the asymptotic form of the scattering amplitude  $f(k, \theta')$  reads

$$f(k, \theta') = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin(\delta_l) P_l[\cos(\theta')], \quad (3)$$

where  $k = \mu v_r / \hbar$ ,  $\theta' \in [0, \pi)$  and  $P_l[\cos(\theta')]$  are the Legendre polynomials. All the complexity of the interatomic potential is contained in the  $k$  dependence of the phase shifts  $\delta_l$ . The differential cross section for nonidentical atoms is given by  $d\sigma_{12}/d\Omega' = |f(k, \theta')|^2$ .

For identical atoms, we have to take into account the (anti)symmetrization principle. Accordingly the differential cross section takes the form

$$\frac{d\sigma_{ii}}{d\Omega'} = |f(k, \theta') + \epsilon f(k, \pi - \theta')|^2, \quad (4)$$

where  $\epsilon=1$  for bosons and  $\epsilon=-1$  for fermions. The integration must be carried out only in the half sphere  $\theta' \in [0, \pi/2)$ . As a consequence of the parity of the spherical harmonic, the only partial waves contributing to the scattering cross section for polarized bosons (respectively, fermions) correspond to even (respectively, odd) values of  $l$ . The interference between partial waves is contained in the differential cross section [16].

A relevant parameter to describe the thermalization of the gases is the collision rate. The average total number of collisions per unit of time  $\Gamma_{ij}$  for atom species  $i$  and  $j$  is obtained by integrating over space and velocity the output channel term of the collision integral in Eq. (2):

$$\Gamma_{ij} = \int d^3r d^3v_A d^3v_B \sigma_{ij} |\mathbf{v}_A - \mathbf{v}_B| f_i(\mathbf{r}, \mathbf{v}_A, t) f_j(\mathbf{r}, \mathbf{v}_B, t). \quad (5)$$

From this quantity we deduce the expression for the mean collision rate per atom  $\gamma_{ij} = \Gamma_{ij}/N_i$ . The integrated cross section  $\sigma_{ij}$  is just the sum of the contributions from partial waves:

$$\sigma_{ij}(v_r) = (1 + \delta_{i,j}) \sum_l \frac{4\pi}{k^2} (2l+1) \sin^2 \delta_l, \quad (6)$$

where the factor  $(1 + \delta_{i,j})$  accounts for the constructive interference of undistinguishable scattering channels for identical atoms,  $\delta_{i,j}$  being the Kronecker delta symbol. The sum in Eq. (6) is taken over even (respectively odd) values for identical ( $i=j$ ) bosons (respectively fermions), and over all integer values for nonidentical atoms ( $i \neq j$ ). The expression for the

collision rate is just a simple sum over the partial waves, it does not exhibit interference. This is to be contrasted with the expression of the thermalization rate as will be illustrated in the next section.

The determination of the thermalization rate is based on the method of averages [17] combined with an appropriate ansatz. Starting from Eq. (1), one readily derives the equation for the average value of a general dynamical quantity  $\mathcal{O}(\mathbf{r}, \mathbf{v}_i)$ :

$$\frac{d\langle \mathcal{O} \rangle_i}{dt} + \langle \mathcal{O} \{f_i, H_i\} \rangle_i = \langle \mathcal{O} I_{ii} \rangle_i + \langle \mathcal{O} I_{ij} \rangle_i, \quad (7)$$

where the average is taken in both position and velocity space:  $\langle \mathcal{O} \rangle_i = \int d^3r d^3v_i f_i(\mathbf{r}, \mathbf{v}_i, t) \mathcal{O}(\mathbf{r}, \mathbf{v}_i) / N_i$ . As a consequence of the collisional invariants — number of atoms, momentum, and energy —  $\langle \mathcal{O} I_{ii} \rangle_i = 0$  if  $\mathcal{O}_i$  is of the form  $\mathcal{O}_i = a(\mathbf{r}) + \mathbf{b}(\mathbf{r}) \cdot \mathbf{v}_i + c(\mathbf{r}) v_i^2$ .

### III. THERMALIZATION IN A BOX

As a first application of the model, we consider the thermalization of a two-species gas in a box of volume  $V$ . We assume that atoms can undergo only specular reflection on the walls of the box, which can be realized experimentally by using the dipolar optical force [18]. Initially, the gases are at different temperatures  $T_1^0 \neq T_2^0$ . After thermalization, the temperature of the gases will be the same  $T_f = (N_1 T_1^0 + N_2 T_2^0) / (N_1 + N_2)$ , within the Boltzmann dynamics. To evaluate the thermalization time, we write the equation for the mean total kinetic energy  $\langle N_i m_i v_i^2 / 2 \rangle_i$  of species  $i$ :

$$\frac{d}{dt} \left\langle \frac{N_i m_i v_i^2}{2} \right\rangle_i = \left\langle \frac{N_i m_i v_i^2}{2} I_{ij} \right\rangle_i \equiv \Sigma_i, \quad j \neq i. \quad (8)$$

The Poisson bracket of Eq. (7) vanishes for a homogeneous system, and the self-collision integral term does not contribute because of conservation of kinetic energy for elastic collisions. The calculation of the collision integral term requires an ansatz for the distribution function. We choose a Gaussian ansatz for the phase-space distribution function of each species:

$$f_i(\mathbf{v}_i, t) = \mathcal{N}_i \exp\left(-\frac{m_i v_i^2}{2k_B T_i}\right), \quad (9)$$

where  $\mathcal{N}_i = N_i m_i^{3/2} / [V(2\pi k_B T_i)^{3/2}]$  is the normalization factor. The time dependence is contained only in the effective temperatures  $T_i$ . We assume that  $(T_1^0 - T_f) / T_f \ll 1$  and  $(T_2^0 - T_f) / T_f \ll 1$ . Consequently, we can evaluate the total number of cross collisions per unit of time by setting  $T_1 = T_2 = T_f$  in Eq. (5) with the Gaussian ansatz for the distribution functions:

$$\Gamma_{12} = \frac{2}{\sqrt{\pi}} \frac{N_1 N_2}{V} c \langle \sigma_{12} \rangle, \quad (10)$$

where we have introduced the velocity  $c = (2k_B T_f / \mu)^{1/2}$  and the thermally averaged cross section  $\langle \sigma_{12} \rangle = 2 \int_0^\infty dx \sigma_{12}(cx) x^3 e^{-x^2}$ .

The expansion around the final temperature  $T_f$  of the collision terms yields a set of linear equations. We can rewrite them in the form (see Appendix A):

$$\frac{d(T_i - T_j)}{dt} = - \left( \frac{N_i + N_j}{N_i N_j} \right) \frac{T_i - T_j}{\tau}, \quad i \neq j, \quad (11)$$

with

$$\frac{1}{\tau} = \frac{8\mu}{3M} \frac{\langle\langle\sigma_{12}\rangle\rangle}{\langle\sigma_{12}\rangle} \Gamma_{12}, \quad (12)$$

where  $\langle\langle\sigma_{12}\rangle\rangle = \int_0^\infty dx \tilde{\sigma}_{12}(cx) x^5 e^{-x^2}$ . The quantity  $\tilde{\sigma}_{12}(cx)$  is defined as

$$\tilde{\sigma}_{12}(cx) = 2\pi \int_0^\pi \sin \theta' (1 - \cos \theta') \frac{d\sigma_{12}}{d\Omega'} d\theta'. \quad (13)$$

The interesting feature of this expression arises from the interference between partial waves resulting from the angular integration. This is in contrast to the expression for the total cross section, which just contains a simple sum over partial waves. We show in Appendix A that interference occurs only between partial waves differing by at most one unit of angular momentum.

We thus find that the relaxation corresponds to an exponential decay but the thermalization times  $\tau$  is *not* in general proportional to the collision rate because it contains partial wave interferences terms.

At very low temperature where only  $s$  waves contribute to the collision process, no interference can occur and the relaxation time is proportional to the inverse of the collision rate. If the cross section depends on the relative wave vector  $k$  of the collision, the factor of proportionality depends on temperature.

For a cross section of the form  $\sigma_{12}(k) = 4\pi a_{12}^2 / (1 + a_{12}^2 k^2)$ , one finds

$$\langle\sigma_{12}\rangle = 4\pi a_{12}^2 \xi [1 - \xi e^\xi \Gamma(0, \xi)],$$

$$\langle\langle\sigma_{12}\rangle\rangle = 2\pi a_{12}^2 \xi [1 - \xi + \xi^2 e^\xi \Gamma(0, \xi)], \quad (14)$$

where  $\xi = \hbar^2 / (2\mu k_B T_f a_{12}^2)$  and  $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ . In the very low-temperature limit  $\xi \gg 1$ , we have  $\langle\sigma_{12}\rangle = \langle\langle\sigma_{12}\rangle\rangle = 4\pi a_{12}^2$ ; in the unitary limit  $\xi \ll 1$ , we obtain  $\langle\sigma_{12}\rangle = 2\langle\langle\sigma_{12}\rangle\rangle = 2\pi \hbar^2 / (\mu k_B T_f)$ . As a consequence, the number  $(\tau \Gamma_{12})$  of interspecies collisions per atom required to equilibrate the temperature of a two-component system made of atoms of the same mass varies from 1.5 in the very low-temperature limit to 3 in the unitary limit according to Eq. (12).

In the following, we explicitly derive the curves for the thermalization time  $\tau$  and the interspecies collision rate  $\gamma$  for the same number of  $^{87}\text{Rb}$  atoms  $N_1 = N_2 = N$  in two different internal states  $|5S_{1/2}, F=2, m=1\rangle$  and  $|5S_{1/2}, F=1, m=-1\rangle$ . This calculation has been performed by taking into account the first five partial wave contributions for the cross collisions [19]. We plot in Fig. 1 the average cross section  $\langle\langle\sigma_{12}\rangle\rangle$  normalized to its value with the cross section taken at zero energy. The resonance shape results from a  $d$ -wave resonance, which in turn tends to magnify the

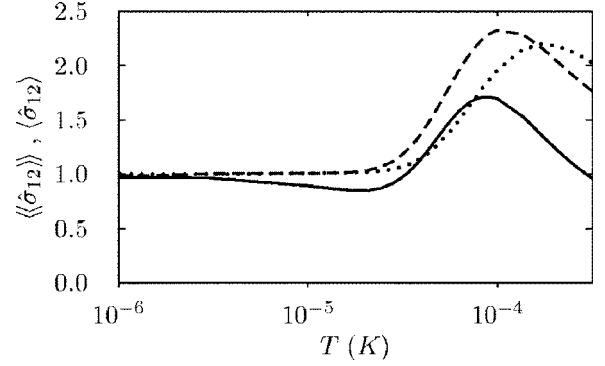


FIG. 1. Averaged cross sections  $\langle\langle\sigma_{12}\rangle\rangle$  and  $\langle\sigma_{12}\rangle$  normalized to their value at zero temperature as a function of the final temperature of the mixture made of rubidium 87 in the two different internal states  $|5S_{1/2}, F=2, m=1\rangle$  and  $|5S_{1/2}, F=1, m=-1\rangle$ . The resonance shape results from a  $d$ -wave resonance. The quantity  $\langle\sigma_{12}\rangle$  is represented in dotted lines. The calculation of  $\langle\langle\sigma_{12}\rangle\rangle$  made with interference terms (solid line) differs notably from the calculation where those terms are neglected (dashed line).

interference terms between  $p$ - $d$ , and  $d$ - $f$  partial waves, leading to a significant difference between the thermalization times calculated with and without the inclusion of interference terms.

#### IV. THERMALIZATION IN AN ISOTROPIC HARMONIC TRAP

We have shown in the previous section that the relation between the collision rate and the thermalization rate is strongly affected by partial wave interferences. In this section we emphasize the role played by the confinement, and we derive the dynamics of the thermalization in all collisional regimes [20].

For the sake of simplicity, we consider the thermalization of an atomic mixture confined in a harmonic and isotropic trap:

$$U_i(x, y, z) = \frac{1}{2} m_i \omega_i^2 (x^2 + y^2 + z^2). \quad (15)$$

By application of the method of averages, we derive the following set of six coupled equations starting from the evolution of the square radius:

$$\frac{d\langle\mathbf{r}^2\rangle_i}{dt} = 2\langle\mathbf{r} \cdot \mathbf{v}_i\rangle_i,$$

$$\frac{d\langle\mathbf{r} \cdot \mathbf{v}_i\rangle_i}{dt} = \langle\mathbf{v}_i^2\rangle_i - \omega_i^2 \langle\mathbf{r}^2\rangle_i + \langle\mathbf{r} \cdot \mathbf{v}_i I_{ij}\rangle_i,$$

$$\frac{d\langle\mathbf{v}_i^2\rangle_i}{dt} = -2\omega_i^2 \langle\mathbf{r} \cdot \mathbf{v}_i\rangle_i + \langle\mathbf{v}_i^2 I_{ij}\rangle_i. \quad (16)$$

The momentum and kinetic energy equations for the individual species include a contribution from the interspecies collision term, expressing the fact that it is the total momentum and total kinetic energy of the system rather than the

individual species' momentum and kinetic energy that are summational invariants. These dissipative contributions to the species equations do not appear in the equation for the radius since the collision is local and the number of atoms for each species is conserved.

In the absence of interspecies interactions the set of Eqs. (16) describes the monopole mode [17,21,22] of both species independently. The collision terms involving  $I_{ii}$  do not contribute to the above equations because all the dynamic quantities are collision invariant for intraspecies collisions. This is why there is no damping for the *breathing* mode of a single-species classical gas confined in a harmonic isotropic trap. In this particular geometry and in the presence of two species, the relaxation only occurs through the interspecies collisions whatever the collisional regime of a given species may be.

The total number of interspecies collisions per unit of time  $\Gamma_{12}$  can be evaluated after the thermalization, once equilibrium has been reached:

$$\Gamma_{12} = \frac{N_1 N_2 c \langle \sigma_{12} \rangle}{\pi^2 \sqrt{2}} \left( \frac{\mu \omega_1 \omega_2 (m_1 + m_2) \omega_1 \omega_2}{k_B T_f m_1 \omega_1^2 + m_2 \omega_2^2} \right)^{3/2}. \quad (17)$$

As the initial state of the mixture is assumed to be not too far from the final state, this expression gives a reliable estimate of  $\Gamma_{12}$  during the thermalization process. The set of Eqs. (16) is exact within the Boltzmann formalism. To calculate quantitatively the relaxation driven by interspecies collisions, we make a Gaussian ansatz for the phase-space distribution function  $f_i \equiv f_i(\mathbf{r}, \mathbf{v}_i, t)$  of each species, with the inclusion of a factor taking into account the space-velocity correlations:

$$f_i = \mathcal{N}_i \exp\left(-\frac{m_i v_i^2 + m_i \omega_i^2 r^2}{2k_B T_i}\right) (1 + \eta_i m_i \mathbf{r} \cdot \mathbf{v}_i), \quad (18)$$

where  $\mathcal{N}_i$  is the normalization factor. The effective temperature  $T_i$  and the parameter  $\eta_i$  are the only time-dependent variables. Such an ansatz is inspired by the exact solution for the phase-space distribution function for the monopole mode, and provides a natural generalization of the local equilibrium distribution. For one species the Gaussian ansatz was shown to be accurate for investigating the damping of the coupled monopole-quadrupole oscillations in an anisotropic harmonic trap from the collisionless to the hydrodynamic regime [17]. At the lowest order, the space-velocity correlations only enter the momentum equations  $\langle \mathbf{r} \cdot \mathbf{v}_i I_{ij} \rangle_i$ , while they can be neglected for the calculation of the term involving the kinetic energy  $\langle \mathbf{v}_i^2 I_{ij} \rangle_i$ . The details of this calculation can be found in Appendix B. We find

$$\langle \mathbf{r} \cdot \mathbf{v}_1 I_{12} \rangle_1 = -\frac{m_1 \omega_1^2 \langle \mathbf{r} \cdot \mathbf{v}_1 \rangle_1 - m_2 \omega_2^2 \langle \mathbf{r} \cdot \mathbf{v}_2 \rangle_2}{m_1 N_1 \omega_1 \omega_2 \tilde{\tau}}, \quad (19)$$

$$\langle \mathbf{v}_1^2 I_{12} \rangle_1 = -\frac{m_1 \langle \mathbf{v}_1^2 \rangle_1 - m_2 \langle \mathbf{v}_2^2 \rangle_2}{m_1 N_1 \tau}, \quad (20)$$

where the time constants  $\tilde{\tau}$  and  $\tau$  are given by

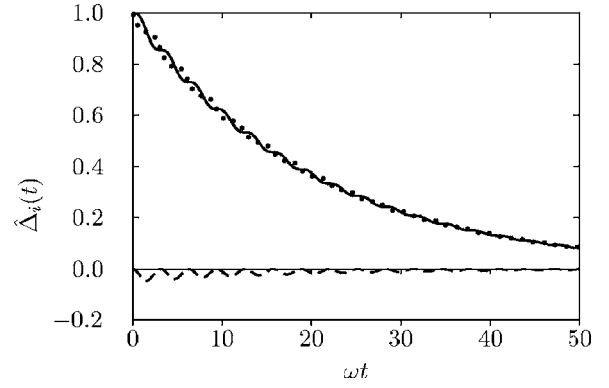


FIG. 2. Evolution of the normalized moments  $\hat{\Delta}_i$  as a function of  $\omega t$  in the collisionless regime  $\omega\tau_0=20$ :  $\hat{\Delta}_1 = \Delta_1(t)/\Delta_1(0)$  (solid line),  $\hat{\Delta}_2 = 2\Delta_2(t)/[\Delta_1(0)+\Delta_3(0)]$  (dashed line), and  $\hat{\Delta}_3 = \Delta_3(t)/\Delta_3(0)$ . They account respectively for the difference in the mean value of the square of the radii, the space-velocity correlations, and the square of the velocity.

$$\frac{1}{\tilde{\tau}} = \frac{4}{3} \frac{\mu \omega_1 \omega_2}{m_1 \omega_1^2 + m_2 \omega_2^2} \frac{\langle \langle \sigma_{12} \rangle \rangle}{\langle \sigma_{12} \rangle} \Gamma_{12}, \quad (21)$$

$$\frac{1}{\tau} = \frac{8}{3} \frac{\mu}{M} \frac{\langle \langle \sigma_{12} \rangle \rangle}{\langle \sigma_{12} \rangle} \Gamma_{12}. \quad (22)$$

Combining Eq. (16), and Eqs. (19)–(22), we obtain a closed set of six linear coupled equations. As a consequence, the relaxation in a trap does not correspond in general to a simple exponential decay.

We demonstrate the physics of rethermalization using a specific example in which  $m_1 = m_2 = m$ ,  $\omega_1 = \omega_2 = \omega$ ,  $N_1 = N_2 = N$  and keeping a constant cross section. However, we emphasize that the conclusions we draw are generic. In order to follow the thermalization it is convenient to introduce the three quantities:

$$\Delta_1(t) = \langle \mathbf{r}^2 \rangle_1(t) - \langle \mathbf{r}^2 \rangle_2(t),$$

$$\Delta_2(t) = \langle \mathbf{r} \cdot \mathbf{v}_1 \rangle_1(t) - \langle \mathbf{r} \cdot \mathbf{v}_2 \rangle_2(t),$$

$$\Delta_3(t) = \langle \mathbf{v}_1^2 \rangle_1(t) - \langle \mathbf{v}_2^2 \rangle_2(t).$$

We characterize the collisional regime through the quantity  $\omega\tau_0$  with  $\tau_0 = N\tau$ . The behavior we obtain is reminiscent of the one of the monopole-quadrupole mode relaxation in a harmonic and anisotropic trap [17]. However, the observables  $\langle \mathbf{r}^2 \rangle_i$ ,  $\langle \mathbf{r} \cdot \mathbf{v}_i \rangle_i$ , and  $\langle \mathbf{v}_i^2 \rangle_i$  and their equation of evolution are not the same. The different regimes depend on the dimensionless parameter  $\omega\tau_0$ .

For  $\omega\tau_0 \gg 1$ , the gas is in the collisionless regime (see Fig. 2). We can recover *formally* the results obtained in the previous section, for a confinement of the gas in a box, by neglecting the space-velocity correlation term in the equation for  $\Delta_3$ , leading to an effective equation for thermalization that is identical to Eq. (11). In presence of the harmonic confinement,  $\Delta_3$  obeys a third-order differential equation according to the set of Eqs. (16). In the collisionless



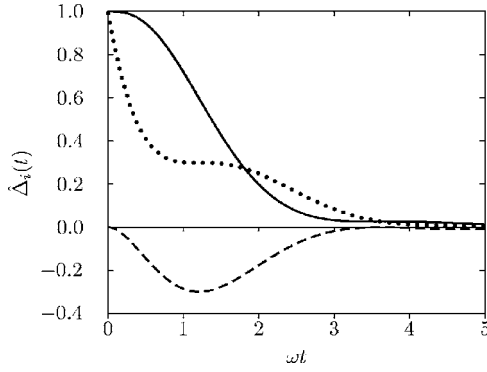


FIG. 3. Evolution of the normalized moments  $\hat{\Delta}_i$  as a function of  $\omega t$  in the intermediate regime  $\omega\tau_0=1$ , same notation as in Fig. 2.

regime, it can be recast, under our assumptions, in the form

$$\frac{d\Delta_3}{dt} \simeq -\frac{\Delta_3}{\tau}.$$

The time needed for thermalization is longer for a confined gas [see for comparison Eq. (11)]. This is in agreement with the results of Refs. [6,7]. This factor of 2 accounts for the fact that the space of configuration is larger for a nonhomogeneous gas, and that the thermalization affects both the space and velocity degrees of freedom.

The decay exhibits oscillations when the collisional regime is such that  $\omega\tau_0 > 0.5$ , with an amplitude scaling as  $1/\omega\tau_0$ . In the intermediate regime ( $\omega\tau_0 \sim 1$ ), the space-velocity correlations play a crucial role in the dynamics, and the decay of the  $\Delta_i$  quantities is not exponential at all as shown in Fig. 3. In the hydrodynamic regime ( $\omega\tau_0 \ll 1$ ), we find two separate time scales: a rapid one for the relaxation in velocity space, and a slow one for the relaxation of the spatial widths (see Fig. 4). We recover in this limit the classical behavior of hydrodynamics, with a short time needed to reach a local equilibrium and a longer time to reach the global equilibrium. The time needed to reach complete thermalization is proportional to  $1/(\omega^2\tau_0)$ . In contrast to the collisionless regime, this time increases with the interspecies collision rate. We have also performed numerical simulations based on molecular dynamics [23] to analyze the rethermal-

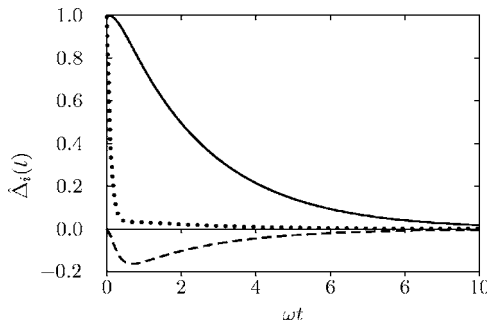


FIG. 4. Evolution of the normalized moments  $\hat{\Delta}_i$  as a function of  $\omega t$  in the hydrodynamic regime  $\omega\tau_0=0.2$ , same notation as in Fig. 2.

ization in all collisional regimes and for arbitrary values of atom numbers, atomic masses, and harmonic trap frequencies for the two species. We have obtained a good agreement with the predictions of our model based on the ansatz (18).

In an anisotropic trap with different trap frequencies  $\omega_i^{(x,y,z)}$ , one can derive a set of equations similar to Eq. (16). In this case the behavior of the relaxation is also influenced by the damping of the quadrupole modes [17].

## V. CONCLUSION

In this paper, we have investigated the thermalization of a nondegenerate atomic mixture. In our analysis we have taken into account both the energy and angular dependence of the interspecies elastic differential cross section. We have derived the explicit form for the thermalization rate, which depends on the interference between the partial waves while the collision rate does not. In addition, we have shown how the dynamics of the thermalization is modified by the collisional regime in presence of a confinement.

## ACKNOWLEDGMENTS

We acknowledge S. Kokkelmans for providing us with the phase shifts in rubidium collisions. We thank D. Roberts for a careful reading of the manuscript. This work was supported by the Délégation Générale de l'Armement (DGA).

## APPENDIX A: COLLISION INTEGRALS CALCULATION IN THE HOMOGENEOUS CASE

The global conservation of kinetic energy in a collision leads to  $\Sigma^+ = \Sigma_1 + \Sigma_2 = 0$ . The symmetries of the collision integral permit one to rewrite the quantity  $\Sigma^- = \Sigma_1 - \Sigma_2$  in the useful form:

$$\Sigma^- = \mu V \int d^3v_0 d^3v_r d^2\Omega' v_r \frac{d\sigma_{12}}{d\Omega'} [\mathbf{v}_0 \cdot (\mathbf{v}_r - \mathbf{v}_r')] \Xi_{1,2}^{1',2'}, \quad (\text{A1})$$

where we have introduced the quantity  $\Xi_{1,2}^{1',2'} = f_1(\mathbf{v}'_1, t) f_1(\mathbf{v}'_2, t) - f_1(\mathbf{v}_1, t) f_1(\mathbf{v}_2, t)$ . The Gaussian ansatz (9) for the distribution functions  $f_i$  allows for an expansion of  $\Xi_{1,2}^{1',2'}$  around the final temperature  $T_f$ . We obtain

$$\Sigma^- = -\frac{\mu^2 V}{k_B T_f} \left( \frac{T_1 - T_2}{T_f} \right) \mathcal{N}_1 \mathcal{N}_2 \int d^3v_0 d^3v_r d^2\Omega' v_r \times \frac{d\sigma_{12}}{d\Omega'} [\mathbf{v}_0 \cdot (\mathbf{v}_r - \mathbf{v}_r')]^2 \exp\left(-\frac{Mv_0^2 + \mu v_R^2}{2k_B T_f}\right). \quad (\text{A2})$$

From the calculation of  $\Sigma^-$ , one obtains the set of linear Eqs. (11) with the explicit form Eq. (12) for the relaxation times  $\tau$ . The angular integration of Eq. (A2) leads to the quantity  $\bar{\sigma}_{12}(cx)$  defined in Eq. (13). From the expansion of the scattering amplitude in terms of partial waves (3), we can recast it in the form

$$\tilde{\sigma}_{12}(cx) = \frac{2\pi}{k^2} \sum_{0 \leq l \leq l' < \infty} \alpha_{l,l'} \sin \delta_l \sin \delta_{l'} \cos(\delta_l - \delta_{l'}), \quad (\text{A3})$$

where we have introduced the dimensionless numerical factors  $\alpha_{l,l'} = (2 - \delta_{l,l'})(2l+1)(2l'+1) \int_{-1}^1 (1-x) P_l(x) P_{l'}(x)$ ,  $\delta_{l,l'}$  being the Kronecker delta symbol. The expression for the coefficients  $\alpha_{l,l'}$  explicitly contains terms describing interference between different partial waves. The properties of the coefficient  $\alpha_{l,l'}$  are the following:  $\alpha_{l,l} > 0$ ,  $\alpha_{l,l \pm 1} < 0$ ,  $\alpha_{l,l'} = 0$  otherwise.

Finally, we perform the Gaussian integration of Eq. (A2), and we work out the expression for the relaxation time  $\tau$  of Eq. (12) according to the notation of Eq. (11).

## APPENDIX B: COLLISION INTEGRALS CALCULATION FOR A HARMONIC CONFINEMENT

The calculation of the quantity  $\langle v^2 I_{12} \rangle$  can be carried out in a way similar to the one used in Appendix A. The terms in

$\eta_i$  of the ansatz (18) do not contribute at the lowest order.

To calculate the quantity  $\langle \mathbf{r} \cdot \mathbf{v}_1 I_{12} \rangle$ , we introduce the quantities  $\Lambda_i = m_i N_i \langle \mathbf{r} \cdot \mathbf{v}_i I_{ij} \rangle_i$  and  $\Lambda^\pm = \Lambda_1 \pm \Lambda_2$ . The global conservation of momentum ensures  $\Lambda^+ = 0$ . The first nonvanishing contribution to the linearized expansion of  $\Lambda^-$  is obtained by setting  $T_1 = T_2 = T_f$  in the ansatz (18). One finds

$$\begin{aligned} \Lambda^- = & -(\eta_1 - \eta_2) \mu^2 \mathcal{N}_1 \mathcal{N}_2 \int d^3 v_0 e^{-M v_0^2 / 2k_B T_f} \\ & \times \int d^3 r d^3 v_r d^2 \Omega' v_r \frac{d\sigma_{12}}{d\Omega'} [\mathbf{r} \cdot (\mathbf{v}_r - \mathbf{v}_r')]^2 \\ & \times \exp\left(-\frac{(m_1 \omega_1^2 + m_2 \omega_2^2) r^2 + \mu v_R^2}{2k_B T_f}\right). \end{aligned} \quad (\text{B1})$$

The ansatz (18) provides the explicit link between the space-velocity correlation moment and the parameters  $\eta_i$ :  $\langle \mathbf{r} \cdot \mathbf{v}_i \rangle_i = 3 \eta_i (k_B T_f)^2 / (m_i \omega_i^2)$ . This expression in combination with Eq. (B1) permits one to derive the explicit expression (21) for the relaxation time  $\tilde{\tau}$  of the space-velocity correlation equation.

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