Dynamics of a classical gas including dissipative and mean-field effects

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By means of a scaling ansatz, we investigate an approximated solution of the Boltzmann-Vlasov equation for a classical gas. Within this framework, we derive the frequencies and the damping of the collective oscillations of a harmonically trapped gas and we investigate its expansion after releasing of the trap. The method is well suited to study the collisional effects taking place in the system and in particular to discuss the crossover between the hydrodynamic and the collisionless regimes. An explicit link between the relaxation times relevant for the damping of the collective oscillations and for the expansion is established.

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The favored signature of Bose-Einstein condensation in weakly interacting gases is the time-of-flight expansion [1]. In this technique, the asymmetric trapping potential is switched off and the evolution of the spatial density is monitored. After a long-time expansion, the observed inversion of the aspect ratio reflects the anisotropy of the initial confinement. In an ideal Bose-Einstein condensate (BEC), this effect is a direct consequence of the Heisenberg uncertainty constraint on the condensate wave function. For an interacting BEC, the inversion is also produced by the anisotropy of the pressure gradients caused by the hydrodynamic forces. The changes in the shape of the expanding gas can be characterized by scaling factors, which provide an easy quantitative tool for the analysis of on-going BEC experiments. The set of equations for those factors have been derived in many papers [2,3]. A similar effect has been predicted also for a Fermi gas in its superfluid phase [4]. A strong anisotropy has been recently measured in the expansion of a highly degenerate Fermi gas [5] close to a Feshbach resonance. Resonance scattering can also give rise to anisotropic expansion in the normal phase as proven in the experiments of Refs. [6,7] carried out in a less degenerate regime. Some recent experiments on bosonic atoms above the critical temperature have also reached the collisional regime investigating both the oscillations of the low-lying quadrupole mode and the expansion in asymmetric traps [8,9].

So far analytic calculations for the expansion of a classical gas have been limited to either the ballistic or to the hydrodynamic regime [3]. It is consequently important to generalize such calculations in all intermediate collisional regimes. This is precisely the main purpose of this paper. We begin by an outline of the theoretical description of the thermal gas based on the Boltzmann-Vlasov equation. Our approach relies on an approximated solution of this equation by means of a scaling ansatz. This solution is used throughout the paper to investigate two kinds of related problems: the lowest collective oscillation modes and the time-of-flight expansion when the confinement is released.

The Boltzmann-Vlasov (BV) kinetic equation for the phase-space distribution $f(t,\mathbf{r},\mathbf{v})$ takes the form

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\frac{1}{m}\frac{\partial(U_{\mathrm{h}}+U_{\mathrm{mf}})}{\partial \mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{v}}=I_{\mathrm{coll}}[f], \tag{1}$$

where $U_{\mathrm{h}}(\mathbf{r})=(m/2)\Sigma_i\omega_i^2 r_i^2$ is the harmonic trapping potential. Interparticle interactions enter Eq. (1) in two different ways [10]. On the one hand, they modify the effective potential through the mean-field term U_{mf} which affects the streaming part of the Boltzmann kinetic equation. The mean-field potential U_{mf} is equal to $2gn$ for bosons and gn for two fermion species [11], where the coupling constant $g=4\pi\hbar^2 a/m$ is fixed by the s-wave scattering length a. The mean-field term is linear in a and is nondissipative. On the other hand, two-body interactions determine the collision integral $I_{\mathrm{coll}}[f]$ which is quadratic in the scattering length and describes dissipative processes. Equation (1) is valid in the semiclassical limit, namely, when the thermal energy is large compared to the separation between the energy eigenvalues of the potential [12,13].

In this paper, we will treat the collision integral within the relaxation-time approximation [12]. This model should suffice to capture the essential physics of the problem. We consequently write

$$I_{\mathrm{coll}}[f]\approx-\frac{f-f_{\mathrm{le}}}{\tau}, \tag{2}$$

where τ is the relaxation time related to the average time between collisions and f_{le} is the local equilibrium density in phase space. As a consequence, f_{le} has a spherical symmetry in velocity space, i.e., it depends on the velocity through $[\mathbf{v}-\mathbf{u}(\mathbf{r})]^2$ where $\mathbf{u}(\mathbf{r})$ is the local velocity field.

The dynamics of the gas will be described by the following scaling ansatz for the nonequilibrium distribution function:

$$f(t,r_i,v_i)=\frac{1}{\prod_j (b_j\theta_j^{1/2})}f_0\left(\frac{r_i}{b_i},\frac{1}{\theta_i^{1/2}}\left(v_i-\frac{\dot{b}_i}{b_i}r_i\right)\right), \tag{3}$$

where f_0 is the equilibrium distribution function which satisfies the equation ($I_{\mathrm{coll}}[f_0]=0$)

$$m\mathbf{v}\cdot\frac{\partial f_0}{\partial \mathbf{r}}=\frac{\partial U_{\mathrm{h}}}{\partial \mathbf{r}}\cdot\frac{\partial f_0}{\partial \mathbf{v}}+\frac{\partial U_{\mathrm{mf}}}{\partial \mathbf{r}}\cdot\frac{\partial f_0}{\partial \mathbf{v}}. \tag{4}$$

The scaling parameter b_i gives the dilation along the ith direction, while θ_i gives the effective temperature in the same direction. The dependence of f on time is contained in the dimensionless scaling parameters b_i and θ_i. Such an ansatz generalizes the one used in Ref. [14]. We recall that in this method the shape of the cloud does not enter explicitly the equations. This is the reason why the solutions are equally valid for a dilute Bose gas above the critical temperature, a dilute Fermi gas in its normal phase, and a classical gas.

Following Ref. [14], one can derive the set of equations for the scaling parameters b_i and θ_i (see Appendix):

$$\ddot{b}_i+\omega_i^2 b_i-\omega_i^2\frac{\theta_i}{b_i}+\omega_i^2\xi\left(\frac{\theta_i}{b_i}-\frac{1}{b_i\prod_j b_j}\right)=0, \tag{5}$$

$$\dot{\theta}_i+2\frac{\dot{b}_i}{b_i}\theta_i=-\frac{1}{\tau}[\theta_i-\bar{\theta}], \tag{6}$$

where the dimensionless parameter $\xi=\langle U_{\mathrm{mf}}\rangle_0/(\langle U_{\mathrm{mf}}\rangle_0+2m\langle v^2\rangle_0/3)$ accounts for the mean-field interaction [15] and $\bar{\theta}=\Sigma_i\theta_i/3$ is the average temperature, for a classical gas $\langle v^2\rangle_0=3k_BT/m$. The parameter ξ is expected to be small for dilute gases $(na^3\ll 1)$ since the ratio U_{mf}/k_BT scales as $(na^3)^{1/3}(n\lambda_{db}^3)^{2/3}$, where λ_{db} is the de Broglie wavelength and n the mean density [14]. Equation (6) shows that the dissipation occurs when the temperature is not isotropic and the relaxation time τ has a finite value.

Equations (5) and (6) are the main results of this paper. The collisionless regime is obtained by taking $\tau_0=\infty$. In this limit, we have the simple relation $\theta_i=b_i^{-2}$ between the scaling parameters, and we recover the equations derived in Ref. [14]. In the opposite limit (hydrodynamic regime) local equilibrium is always ensured because of the high collision rate. As a consequence, we have $\theta_i=\bar{\theta}=\Pi_j b_j^{-2/3}$ and Eqs. (5) and (6) can be recast in the form:

$$\ddot{b}_i+\omega_i^2 b_i-\frac{\omega_i^2}{b_i\prod_j b_j^{2/3}}+\omega_i^2\xi\left(\frac{1}{b_i\prod_j b_j^{2/3}}-\frac{1}{b_i\prod_j b_j}\right)=0. \tag{7}$$

For $\xi=0$ (no mean field) we recover the equations first derived in Ref. [3]. Note that in both the collisionless and the hydrodynamic regimes, the collisional term does not contribute since there is no dissipation in these limits. We next focus our attention on the intermediate regimes where the collision term enters explicitly the equations of motion.

Let us first study the breathing mode in the case of a spherical harmonic trapping with angular frequency ω_0. In this case, we find a solution with $b_i=b$ and $\theta_i=b^{-2}$. For such a solution the collision term identically vanishes in all intermediate collisional regimes. Our approach can be readily generalized to lower dimensions leading to the frequency $\omega_0[4+\xi(d-2)]^{1/2}$ for the monopole mode [14], where d is the dimension of space. In two dimensions the mean field does not affect the frequency of the monopole. This comes out from the fact that in this case the ansatz is an exact solution of the BV equations, as already stressed in Ref. [16].

We now consider a sample of atoms confined in a three-dimensional cylindrically symmetric harmonic potential. We denote the ratio between the axial and radial angular frequencies by $\lambda=\omega_z/\omega_\perp$. Expanding Eqs. (5) and (6) around equilibrium $(b_i=\theta_i=1)$ we get a linear and closed set of equations which can be solved by looking for solutions of the type $e^{i\omega t}$. The associated determinant yields the following dispersion law:

$$\left(A[\omega]+\frac{1}{\tau_0}B[\omega]\right)\left(C[\omega]+\frac{1}{\tau_0}D[\omega]\right)=0, \tag{8}$$

where $A[\omega]=\omega^2(\omega^2-\omega_{\mathrm{cl}+}^2)(\omega^2-\omega_{\mathrm{cl}-}^2)$, $B[\omega]=\omega(\omega^2-\omega_{\mathrm{hd}+}^2)(\omega^2-\omega_{\mathrm{hd}-}^2)$, $C[\omega]=\omega(\omega^2-\omega_{\mathrm{cl}}^2)$, and $D[\omega]=(\omega^2-\omega_{\mathrm{hd}}^2)$ and (cl) and (hd) refer to the collisionless and hydrodynamics regimes, respectively. The coefficient τ_0 is the value of the relaxation time τ calculated at equilibrium and

$$\omega_{\mathrm{cl}\pm}^2=\frac{\omega_\perp^2}{2}[4(1+\lambda^2)-\lambda^2\xi \pm\sqrt{16+\lambda^4(4-\xi)^2+8\lambda^2(\xi^2-4+\xi)}],$$

$$\omega_{\mathrm{cl}}^2=\omega_\perp^2(4-2\xi),$$

$$\omega_{\mathrm{hd}\pm}^2=\frac{\omega_\perp^2}{3}\left[5+4\lambda^2+\xi(1+\lambda^2/2) \pm\frac{1}{2}\sqrt{(10+8\lambda^2+2\xi+\lambda^2\xi)^2-72\lambda^2(4+\xi)}\right],$$

$$\omega_{\mathrm{hd}}^2=2\omega_\perp^2.$$

Here ξ is the parameter accounting for the mean-field effects. The solution of Eq. (8) interpolates the frequencies of the low-lying modes for all collisional regimes ranging from the collisionless to the hydrodynamic one. As the confinement is cylindrically symmetric around the z axis, we can label the modes by their angular azimuthal number M. The first factor of the left-hand side of Eq. (8) gives the frequencies of the two $M=0$ modes, while the second factor gives that of the quadrupole $(M=\pm 2)$ mode. The roots of A and C have already been obtained in Ref. [14], and correspond to the frequencies of the low-lying modes of a collisionless gas in the presence of mean field. Equation (8) for $\xi=0$ has been derived in Ref. [17] and the corresponding frequencies have been investigated experimentally [8]. For $\xi=1$, corresponding to $\langle gn_0\rangle_0 \gg \langle v^2\rangle_0$, we find $\omega^2_{\mathrm{cl}\pm}=\omega^2_{\mathrm{hd}\pm}$, $\omega^2_{\mathrm{cl}}=\omega^2_{\mathrm{hd}}$, and the frequencies coincide with the ones predicted for a Bose-Einstein condensate in the Thomas-Fermi regime [18].

So far, we have not given the explicit link between the relaxation time entering Eq. (8) and the collision rate. Following Ref. [17], we can establish this link for a classical gas by means of a Gaussian ansatz for the equilibrium distribution function $f(\mathbf{r},\mathbf{v},t)$. One obtains $\tau_0=5/(4\gamma)$ where $\gamma=2(2\pi)^{-1/2}n_{\max}\sigma v_{\mathrm{th}}$ is the classical collision rate where $v_{\mathrm{th}}=(k_BT/m)^{1/2}$ is the thermal velocity, $n_{\max}$ is the peak density, and σ is the cross section which is assumed to be velocity independent. For bosons the link between the scattering length and the cross section is $\sigma=8\pi a^2$ whereas for two fermion species one has $\sigma=4\pi a^2$.

We now establish the set of equations that describe the time-of-flight expansion. In the collisionless regime where the mean free path is very large with respect to the size of the trapped cloud and in the absence of mean-field contribution, we readily obtain the exact equations $\ddot{b}_i=\omega_i^2/b_i^3$ which admit the solutions $b_i(t)=(1+\omega_i^2t^2)^{1/2}$, leading to an isotropic density and velocity distributions after a long-time expansion.

When the effect of collisions is important the physics of the expansion changes dramatically. As an example, the radial directions of a cigar-shaped cloud expand faster than the longitudinal one, resulting in a final anisotropic velocity distribution. So far, an analytic approach has been proposed only in the full hydrodynamic regime [3]. However, this approach assumes that the hydrodynamic equations are always valid during the expansion. In general, this cannot be the case since the density decreases during the expansion, reducing the effect of collisions. Alternatively, the expansion of an interacting Bose gas above T_c has been investigated by means of Monte Carlo simulations [19].

In our approach, we provide an interpolation between the two opposite collisionless and hydrodynamic regimes using the scaling formalism. The decrease of the collision rate during the expansion yields a nonconstant relaxation time $\tau(b_i,\theta_i)$ that depends explicitly on the scaling parameters reflecting the changes of the density and the temperature during the expansion. As a result, the expansion is described by the following set of six nonlinear equations:

$$\ddot{b}_i-\omega_i^2\frac{\theta_i}{b_i}+\omega_i^2\xi\left(\frac{\theta_i}{b_i}-\frac{1}{b_i\prod_j b_j}\right)=0,$$

$$\dot{\theta}_i+2\frac{\dot{b}_i}{b_i}\theta_i=-\frac{1}{\tau(b_i,\theta_i)}\left(\theta_i-\frac{1}{3}\sum_j \theta_j\right). \tag{9}$$

The dependence of the relaxation time τ on the scaling parameters is obtained by noting that the collision rate γ scales as $nT^{1/2}$. Using the scaling transformation $n\to n_0(\Pi_j b_j)^{-1}$ and $T\to T_0\overline{\theta}$, where n_0 and T_0 are the initial density and temperature, respectively, we deduce

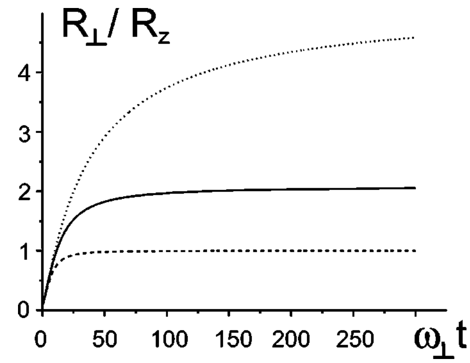

FIG. 1. Aspect ratio as a function of the normalized time $\omega_\perp t$ for different collisional regimes (initial aspect ratio $\lambda=0.1$): collisionless (dashed line, $\tau_0\to\infty$), intermediate collisional regime (solid line, $\omega_\perp\tau_0=0.1$), and hydrodynamic regime (dotted line, $\tau_0\to 0$).

$$\tau(b_i,\theta_i)=\tau_0\left(\prod_j b_j\right)\left(\frac{1}{3}\sum_k \theta_k\right)^{-1/2}, \tag{10}$$

where τ_0 is the average time of collisions at equilibrium [20]. Since both results (8) for the dispersion of the linear oscillations and Eqs. (9) and (10) for the expansion have been derived starting from the same scaling equations (5) and (6), the relaxation time τ_0 entering the two processes is the same. As a consequence, the combined investigation of the expansion and of the quadrupole oscillations can provide a useful check of consistency of the approach and, possibly, useful constraints on the value of the cross section.

The time evolution of the aspect ratio $R_\perp(t)/R_z(t)=\lambda b_\perp(t)/b_z(t)$ in the absence of the mean field is depicted in Fig. 1 for different values of the product $\omega_\perp\tau_0$ and for an initial aspect ratio $\lambda=0.1$. In the collisionless regime ($\tau_0=\infty$) the aspect ratio tends asymptotically to unity, reflecting the isotropy of the initial velocity distribution. For other collisional regimes, the asymptotic aspect ratio is larger than 1 as a consequence of collisions during the expansion. We find a continuous transition from the collisionless to the hydrodynamic prediction as we decrease τ_0 from infinity to zero. We then conclude that in general the expansion cannot be described with either the hydrodynamic or the collisionless prediction [21], but requires a full solution of our equations (9). Similar conditions have been already encountered experimentally for classical or almost classical gases [9]. We also note that it is very important to take into account the time dependence of the relaxation time, accounted for by the scaling law (10). For example, by simply using $\tau=\tau_0$ during the whole expansion, the curve $\omega_\perp\tau=0.1$ of Fig. 1 (solid line) would be shifted upward and the resulting prediction would be much closer to the hydrodynamic curve (dotted line).

Let us finally comment on the effect of quantum statistics on the calculation of the relaxation time. For a Bose gas at temperature above T_c the problem has been investigated in Ref. [22] where it has been shown that statistical effects do not play a significant role (see also Ref. [23]). In contrast, the relaxation time in a harmonically trapped dilute Fermi gas has been shown to be strongly affected by Pauli blocking at low temperature [24]. The effects of collisions in a strongly interacting Fermi gas, including the unitarity limit, have been recently addressed in Ref. [25].

In conclusion, we have provided a generalization of the scaling approach to the dynamics of dilute gases by including the effects of collisions. This generalization is expected to be important in view of the possibility of tuning the scattering length using Feshbach resonances as well as for an accurate thermometry of the gas after expansion. Former applications have been already reported in Ref. [23].

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APPENDIX: EQUATIONS

We define the following ansatz for the nonequilibrium distribution function: $f(\mathbf{r},\mathbf{v},t)=\Gamma f_0(\mathbf{R}(t),\mathbf{V}(t))$ with $R_i=r_i/b_i$, $V_i=(v_i-\dot{b}_i r_i/b_i)\theta_i^{-1/2}$, and $\Gamma=\Pi_j b_j^{-1}\theta_j^{-1/2}$. The dependence on time is contained in the parameters b_i and θ_i. Following Ref. [14], we substitute this ansatz into Eq. (4) and use the equation for the equilibrium distribution f_0. We find that

$$\dot{\Gamma} f_0+\Gamma\sum_i\left\{V_i\frac{\partial f_0}{\partial R_i}\left(\frac{\theta_i^{1/2}}{b_i}-\frac{1}{b_i\theta_i^{1/2}}\frac{1}{\Pi_j b_j}\right)-\frac{\partial f_0}{\partial V_i}\left[\frac{R_i}{\theta_i^{1/2}}\left(\ddot{b}_i+\omega_i^2 b_i-\frac{\omega_i^2}{b_i}\frac{1}{\Pi_j b_j}\right)+V_i\left(\frac{1}{2}\frac{\dot{\theta}_i}{\theta_i}+\frac{\dot{b}_i}{b_i}\right)\right]\right\}=I_{\mathrm{coll}}. \tag{A1}$$

Performing the integration in phase space, we calculate the average moment of R_iV_i, namely, $\int R_iV_i$[Eq. (A1)] d^3Rd^3V/N. This leads to Eq. (5). Note that this equation is not affected by the collision integral since the quantity term R_iV_i is conserved by collisions.

To derive Eq. (6), we consider instead the average moment of V_i^2. This yields

$$\frac{\dot{\theta}_i}{\theta_i}+2\frac{\dot{b}_i}{b_i}=\frac{m}{N\Gamma k_B T_0}\int V_i^2 I_{\mathrm{coll}} d^3R d^3V, \tag{A2}$$

where T_0 is the equilibrium temperature. Different from Eq. (5), Eq. (A2) depends explicitly on the collision integral. In order to calculate the right-hand side of Eq. (A2), we use the relaxation time-approximation $I=-(f-f_{\mathrm{le}})/\tau_0$. The first term gives $\int V_i^2 f d^3Rd^3V=N\Gamma k_B T_0/m$. To obtain a relation among the temperature-scaling parameters one uses the identity $\langle v^2\rangle=\langle v^2\rangle_{\mathrm{le}}$, from which we deduce that $\overline{\theta}=\Sigma_i\theta_i/3$ emerges to be the average temperature of the sample. The contribution of the local equilibrium term to the second term is obtained noting that, at local equilibrium, $\theta_i^{\mathrm{le}}=\overline{\theta}$: $\int V_i^2 f_{\mathrm{le}}d^3Rd^3V=\overline{\Gamma}\int V_i^2 f_0(\overline{\mathbf{R}},\overline{\mathbf{V}})d^3Rd^3V=N\Gamma\overline{\theta}k_BT_0/(m\theta_i)$. Hence Eq. (A2) can be recast in the form (6).

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