Mean-field effects in a trapped gas

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In this paper I investigate mean-field effects for a Bose gas harmonically trapped above the quantum transition temperature in the collisionless regime. I point out that these effects can play a role in low dimensional systems. My treatment relies on the Boltzmann equation with the inclusion of the mean-field term. I first discuss the equilibrium state then derive the dispersion relation for collective oscillations (monopole, quadrupole, and dipole modes) in D dimensions.

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The dynamics of Bose-Einstein condensates (BECs) of dilute atomic gases are described by the Gross-Pitaevskii equation ([1] and references therein). The main feature of this equation is the mean-field term arising from the interaction between particles. Most of the BEC experiments are carried out in the Thomas-Fermi regime where the interaction energy dominates the kinetic energy, resulting in an inverted parabola shape of the condensate density. However, the mean-field term in a Bose gas is not only found well below its critical temperature T_c, but also has a contribution above T_c, and is even magnified by a kind of Hanbury-Brown and Twiss factor. Up to now most BEC experiments have been performed in the collisionless regime (i.e., the mean free collision rate is small with respect to the trap frequencies) where the contribution of the mean-field of noncondensed atoms is negligible.

In order to specify the role of dimensionality, I introduce the dimensionless ratio of the mean-field energy to the thermal energy $\zeta = gn/k_B T$, where n is the density, g accounts for the mean-field strength in D dimensions, and T is the temperature. In three dimensions (3D) the strength of the pseudopotential is $g = 4\pi\hbar^2 a/m$ where a is the s-wave scattering length that replaces the true two-body potential at low energies. One readily establishes for 3D that $\zeta_{3D} \sim (na^3)^{1/3}(n\lambda_{dB}^3)^{2/3}$, where $\lambda_{dB} = h(2\pi m k_B T)^{-1/2}$ is the de Broglie wavelength. Consequently, for a dilute Bose gas above the critical temperature $\zeta_{3D} \ll 1$. Thus the results presented in this paper are valid as corrections in 3D. In two dimensions (2D) or in quasi-2D [2] (3D but with one "frozen" direction in which particles undergo zero point oscillations) and in the weakly interacting limit, the quantity ζ_{2D} is only logarithmically small with respect to λ_{dB}^2 and the mean-field energy can be comparable to the thermal energy above the quantum transition temperature (Kosterlitz-Thouless [3]). In this paper, the 2D coupling constant g is taken constant, I consequently neglect the logarithmical dependence on the density. In one dimension (1D) or in quasi-1D [4], the quantity ζ_{1D} is of the order of $(n\lambda_{dB})^2(nl_c)^{-2}$ where the correlation length is defined by $l_c = \hbar/\sqrt{mgn}$. Classical description and mean-field theory can be used up to the regime where $\lambda_{dB} \sim 1/n \sim l_c$, i.e., $\zeta_{1D} \sim 1$. In the regime where l_c is much smaller than the mean interparticle separation the gas acquires Fermi properties and is then called a gas of impenetrable bosons or Tonks gas [5].

Finally, in low dimensional systems, the role of the mean-field even above the critical temperature of a quantum transition is more important. Some experiments have prepared low dimensional condensates in optical and magnetic traps [6]. Another class of experiments [7] has been done in the noncondensed regime with the same kind of confinement. Hydrogen atoms on liquid ^{4}He also provide a two-dimensional system [8]. Experiments performed on microchips offer the possibility of investigating the low dimensional regime [9,10].

In this paper my aim is to extend the traditional treatment of the Bose gas above the critical temperature by taking into account the mean-field contribution of particle interactions. The method consists of including the classical mean-field term, also known as the Vlasov contribution, in the Boltzmann equation. So far, the collective oscillations of a Bose gas above the critical temperature have been investigated without the mean-field contribution in the hydrodynamic regime [11,12], and an interpolation formula from the collisionless up to the hydrodynamic regime has been proposed [13,14].

In Sec. I, we briefly recall the general framework based on the Boltzmann equation. The stationary solution is discussed in Sec. II. In Sec. III, we derive the frequencies of the low lying modes of a Bose gas for positive and negative scattering length by means of a scaling ansatz. We obtain an interpolation formula from the collisionless gas to the interaction-dominated thermal gas (Vlasov gas) in the absence of dissipation.

I. FORMULATION

In traditional BEC experiments, the Bose gas above the critical temperature is well described by the classical Boltzmann equation [15], or the Uhlenbeck-Boltzmann equation [16] if experiments are sufficiently accurate to measure the deviation from the classical distribution. In the following, I present the method for generalizing this equation when the mean-field contribution is included.

I consider an ensemble of harmonically trapped thermal

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atoms that evolves according to the Boltzmann-Vlasov kinetic equation [15,17]:

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\frac{\partial U}{\partial \mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{v}}-2\frac{g}{m}\frac{\partial n}{\partial \mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{v}}=I_{\rm coll}, \qquad (1)$$

where $f(\mathbf{r},\mathbf{v},t)$ is the single particle phase space distribution function, $U=\Sigma_i\omega_i^2 r_i^2/2$ the confining potential, $n=\int f d^D v$ the density, and g the mean-field strength in D dimensions. $I_{\rm coll}$ is the collisional integral describing relaxation processes. Note that $I_{\rm coll}=0$ in 1D because of conserved quantities. The Vlasov term [last term of the left-hand side of Eq. (1)], a Hartree-Fock mean-field term [18], is even magnified for point-like interactions by a factor of 2 with respect to the condensate for the same density. Indeed, for noncondensed clouds both the Hartree and the Fock terms contribute, whereas for the condensate only the former contributes.

The kinetic equation (1) is valid for $k_BT\gg\hbar\omega$ where ω is the typical trap frequency and for $a\ll a_h$ where $a_h=[\hbar/(m\omega)]^{1/2}$ is the oscillator quantum length. One must check that the s-wave approximation is valid. For instance, it requires that $a\ll\lambda_{dB}$ in 3D.

II. EQUILIBRIUM STATE

In equilibrium, Eq. (1) reads

$$\sum_{i=1}^{D}\left(v_i\frac{\partial f_0}{\partial r_i}-\omega_i^2 r_i\frac{\partial f_0}{\partial v_i}-\frac{2g}{m}\frac{\partial n_0}{\partial r_i}\cdot\frac{\partial f_0}{\partial v_i}\right)=0. \qquad (2)$$

By multiplying Eq. (2) by $v_j r_j$ and integrating over space and velocity, I deduce the average size $\langle r_j^2\rangle$ along the j axis [13]:

$$\omega_j^2\langle r_j^2\rangle-\langle v_j^2\rangle-\frac{g}{mN}\int n_0^2 d^D r=0. \qquad (3)$$

As expected, repulsive interactions ($g>0$) favor a reduction of the density from the free particle case. The opposite behavior is obtained in the case of attractive interactions ($g<0$). We can extract the shape of the density by searching for a factorized solution of Eq. (2) of the form $f_0(\mathbf{r},\mathbf{v})=n_v(\mathbf{v})n_0(\mathbf{r})$. I find a Gaussian spherical distribution for the velocity. The density distribution is a solution of the following equation:

$$\kappa\ln(n_0)+2gn_0/m=\mu-\sum_j\omega_j^2 r_j^2/2, \qquad (4)$$

where $\kappa=\int v_j^2 n_v d^D v=k_BT/m$. In two limiting cases, the solution has a simple form. For $g=0$, I find the Gaussian shape as expected for a harmonic confinement without the Vlasov term. On the contrary, in the limit where the interparticle interactions also dominate and are repulsive, the shape of the cloud is determined by a balance between the harmonic oscillator and the interaction energy resulting in an approximately inverted parabola. This is the same shape as found for a harmonically trapped BEC in the Thomas-Fermi regime [1], since in this case the mean-field term also dominates. This result can also be shown in the classical hydrodynamic regime [12] under the same conditions. Strictly speaking this situation can be approached only for low dimensional systems. For intermediate g, Eq. (4) gives the proper interpolation between the Gaussian and the Thomas-Fermi shape.

For $g<0$, the density distribution is sharpened with respect to the free Gaussian one. In fact, by increasing the number of atoms, the spatial extent of the distribution is reduced. If the attractive energy overwhelms the kinetic energy the cloud collapses. One may obtain the criterion for such a collapse in 3D by means of a Gaussian ansatz [19] and finds $a_c=33a_hN^{-1}(a_h/\lambda_{dB})^5$. However, this result is out of the range of validity of the classical approximation.

III. COLLECTIVE OSCILLATIONS OF A COLLISIONLESS GAS

In this section I investigate the collective oscillations of a Vlasov gas, i.e., in the absence of the dissipative term ($I_{\rm coll}$) but with the mean-field contribution.

A. Scaling ansatz method

I study the low-lying modes (monopole, quadupole, and dipole) by means of the scaling factor method [12,20–23] in D dimensions. I recall that in this method the proper shape of the cloud does not enter directly in the equations. This is why the solutions are equally valid for a Bose gas just above the critical temperature as for a classical gas. I make the following ansatz for the nonequilibrium distribution function: $f(\mathbf{r},\mathbf{v},t)=f_0(\mathbf{R}(t),\mathbf{V}(t))$ with $R_i=r_i/\lambda_i$ and $V_i=\lambda_i v_i-\dot{\lambda}_i r_i$, so that the density profile is implicitly included in this ansatz. The dependence on t is contained in the free parameters λ_i. By substituing this ansatz into Eq. (1), I find

$$\sum_i\left\{\frac{V_i}{\lambda_i^2}\frac{\partial f_0}{\partial R_i}-\lambda_i R_i(\ddot{\lambda}_i+\omega_i^2\lambda_i)\frac{\partial f_0}{\partial V_i}-\frac{2g}{\Pi_j\lambda_j}\frac{\partial n_0}{\partial R_i}\frac{\partial f_0}{\partial V_i}\right\}\simeq 0. \qquad (5)$$

This equation can be combined with Eq. (2) taken at the phase space point ($\mathbf{r}=\mathbf{R},\mathbf{v}=\mathbf{V}$) in order to replace the last term of Eq. (5) by a linear superposition of $\partial f_0/\partial R_i$ and $\partial f_0/\partial V_i$. I finally obtain

$$\sum_i\left\{\left(\frac{V_i}{\lambda_i^2}-\frac{V_i}{\Pi_j\lambda_j}\right)\frac{\partial f_0}{\partial R_i}-\lambda_i R_i\left(\ddot{\lambda}_i+\omega_i^2\lambda_i-\frac{\omega_i^2}{\lambda_i\Pi_j\lambda_j}\right)\frac{\partial f_0}{\partial V_i}\right\}=0. \qquad (6)$$

This equation provides the constraints on the ansatz. The first average moment [13] of R_iV_i, namely $\int R_iV_i[\cdots]d^DRd^DV/N$, where $[\cdots]$ represents Eq. (6) leads to a set of Newton-like second order ordinary differential equations:

$$\ddot{\lambda}_i+\omega_i^2\lambda_i-\frac{\omega_i^2}{\lambda_i^3}+\omega_i^2\xi\left(\frac{1}{\lambda_i^3}-\frac{1}{\lambda_i\Pi_j\lambda_j}\right)=0 \qquad (7)$$

with $\xi=g\langle n_0\rangle/(g\langle n_0\rangle+k_BT)$. Expanding Eq. (7) around the equilibrium value defined by $\lambda_i=1$, I find the frequencies of the low energy collective oscillations of a Bose gas. This expansion allows the determination of the "quadratic modes," namely the monopole and the quadrupole modes.

B. Monopole mode

Consider the case of isotropic harmonic confinement $\omega_i = \omega_0$. Then the scaling ansatz is in the kernel of the collision integral and provides a solution of Eq. (1) which is also in the collisional regime. An exact solution of this mode was first reported in [24] for the classical Boltzmann equation without mean-field.

The small amplitude expansion of Eq. (7) gives the frequency of the monopole mode (also called the breathing mode), $\omega_0\sqrt{4+\xi(D-2)}$, for all dimensions and in the presence of the total effects of collisions (mean-field term and dissipation via $I_{\rm coll}$). In 3D this frequency ranges from $2\omega_0$ in the absence of the mean-field term up to $\sqrt{5}\omega_0$ when the mean-field dominates. In the latter case one obtains the same result as expected for a Bose-Einstein condensate in the Thomas-Fermi limit [25]. In 2D I find $2\omega_0$ for the monopole mode, a result independent of the mean-field which is a special feature of 2D already investigated in Ref. [26]. In this case, the scaling ansatz provides an exact solution of Eq. (1). In 1D, the monopole frequency ranges from $2\omega_0$ to $\sqrt{3}\omega_0$ when the mean-field dominates. The latter has been derived in [27] in the context of a 1D Bose-Einstein condensate. The same frequency for the monopole mode is also obtained for a very different system: trapped ions in 1D [28]. However, it is a coincidence that happens only for the Coulomb potential.

C. Quadrupolar mode

In 2D and 3D, Eq. (7) provides the mean-field contribution to quadrupolar collective oscillations for a collisionless gas. In 2D, Eq. (7) gives two coupled equations for λ_x and λ_y, which, after linearization, yield the dispersion relation

$$\omega^2=\frac{1}{2}[(4-\xi)(\omega_x^2+\omega_y^2) \pm[(4-\xi)^2(\omega_x^2-\omega_y^2)^2+4\omega_x^2\omega_y^2\xi^2]^{1/2}]. \quad (8)$$

For $\xi=0$, I recover the single particle excitation frequency of the cloud: $\omega=2\omega_i$ for each spatial direction [29]. In the limit $\xi=1$, this relation can be derived from a purely hydrodynamic approach [11,12] by taking into account the mean-field contribution in the same limit. Formula (8) also provides finite temperature corrections to this regime and the proper interpolation inbetween these two limiting cases.

For a cylindrical 3D harmonic trap around the z axis (I denote $\beta=\omega_z/\omega_\perp$), I can label the modes by their angular azimuthal number M, since the angular momentum along the z axis is conserved. I find the eigenfrequencies of the mode $M=0$ (coupling between quadrupole and monopole modes):

$$\omega^2=\frac{\omega_\perp}{2}[4+4\beta^2-\xi\pm([4+4\beta^2-\xi]^2+8\beta^2[-8+2\xi+\xi^2])^{1/2}], \quad (9)$$

and the frequency of the quadrupole mode with azimuthal quantum number $M=2$: $\omega^2/\omega_\perp^2=2(2-\xi)$. The limit $\xi\sim1$ gives the formulas derived by Stringari [25] for the low energy excitation spectrum of a BEC in the limit where the energy of interaction predominates over the kinetic energy. Low-lying collective excitations (monopole, quadrupole) spectrum of a BEC are rather a proof of mean-field dominated physics than a direct proof of superfluidity in the Landau sense [30]. The low-lying shape oscillations of noncondensed clouds have already been observed experimentally [31,32], but not in the regime where the mean-field can play a detectable role. As already pointed out, my calculation in 3D is valid only if $\xi\ll1$. For an isotropic 3D trap ($\omega_i=\omega_0$), the oscillation frequency splits into the monopole mode with a frequency $\omega_M\simeq2\omega_0(1+\xi/8)$ and the quadrupole mode with a frequency $\omega_Q\simeq2\omega_0(1-\xi/4)$ as soon as I take into account the mean-field.

D. Dipolar mode

The dipolar mode which corresponds to the rigid motion of the density profile is not affected by the mean-field. This can be shown by searching for a solution of the form $f(\mathbf{r},\mathbf{v},t)=f_0(\mathbf{R}(t),\mathbf{V}(t))$ with $R_i=r_i-\eta_i$ and $V_i=v_i-\dot{\eta}_i$. Each component η_i is time dependent. Following the same procedure, I readily establish the equation of motion for η by taking the average value of V_i: $\ddot{\eta}_i+\omega_i^2\eta_i=0$. I recover the fact that Kohn modes do not depend on interactions. This result is naturally unchanged if I take into account the collisional integral contribution.

E. Discussion

$I_{\rm coll}=0$ is strictly speaking only applicable for a collisionless gas ($a\to0$) and to the hydrodynamic regime (collision rate $\gg$ trap frequencies). In between, $I_{\rm coll}$ is negligible with respect to the Vlasov term if $a\ll\tilde{a}$, where $\tilde{a}$ is a critical value for the scattering length. The collision integral is of the same order of magnitude as the Vlasov contribution when $g=\sigma v^2\ell$, where $\sigma=8\pi a^2$ is the elastic cross section, v is a typical thermal velocity, and ℓ a typical size. I take $\ell\sim v/\omega$ and find $\tilde{a}=0.03\lambda_{dB}(\lambda_{dB}/a_h)^2$. Just above the critical temperature, $\tilde{a}$ is of the order of the scattering length for the experiment of Ref. [9] performed on a microchip. For the metastable helium experiment [33] $\tilde{a}$ is slightly smaller than the scattering length. Even though these experiments are not in the Thomas-Fermi regime, one can no longer ignore mean-field effects. In many BEC experiments, $\zeta_{max}<10^{-4}$ which clearly justifies that it is neglected. However, in Ref. [9] this ratio is of the order of $\zeta_{max}\sim10\%$ and could be increased by stronger longitudinal confinement. Further experiments on microchips [10] should allow for the reduction of dimensionality. Dipolar traps can also be a useful tool [7]. These techniques should help for the observation of classical mean-field above a quantum transition. In [33], the quantity ζ_{max} is of the order of $20\pm10\%$ as a consequence of the large value of the scattering length and the high density of the sample.

Two-dimensional quasi-condensates have been recently observed in a gas of hydrogen atoms on a liquid ^{4}He surface [8]. The regime investigated in this experiment corresponds to a mean field energy of the order of the thermal energy ($\zeta_{max}\sim 1$). This kind of system may be very well suited for studying the effects that I present in this paper.

IV. CONCLUSIONS

Mean-field effects for a Bose gas above the quantum transition temperature play an increasing role as the dimensionality is reduced. This paper deals with the contribution of the mean-field to the low-lying collective modes of such a gas. I derive, even for an interacting gas, the frequency of the monopole mode under isotropic harmonic confinement. Note that results derived herein hold also for the in phase motion of the two-component Fermi-system when the mean-field plays a role.

The mean-field contribution could be seen directly on the equilibrium shape of the gas *in situ*. Time-of-flight measurement may allow for the direct observation of its contribution.

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[1] F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[2] D.S. Petrov, M. Holzmann, and G.V. Shlyapnikov, Phys. Rev. Lett. **84**, 2551 (2000); D.S. Petrov and G.V. Shlyapnikov, Phys. Rev. A **64**, 012706 (2001).
[3] J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973); J.M. Kosterlitz, *ibid.* **7**, 1046 (1974).
[4] D.S. Petrov, G.V. Shlyapnikov, and J.T.M. Walraven, Phys. Rev. Lett. **85**, 3745 (2000).
[5] E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963); E.H. Lieb, *ibid.* **130**, 1616 (1963); M. Girardeau, J. Math. Phys. **1**, 516 (1960); L. Tonks, Phys. Rev. **50**, 955 (1936).
[6] A. Gorlitz *et al.*, Phys. Rev. Lett. **87**, 130402 (2001); F. Schreck *et al.*, *ibid.* **87**, 080403 (2001); S. Burger *et al.*, Europhys. Lett. **57**, 1 (2002).
[7] V. Vuletic, C. Chin, A.J. Kerman, and S. Chu, Phys. Rev. Lett. **81**, 5768 (1998); V. Vuletic, A.J. Kerman, C. Chin, and S. Chu, *ibid.* **82**, 1406 (1999); I. Bouchoule, H. Perrin, A. Kuhn, M. Morinaga, and C. Salomon, Phys. Rev. A **59**, R8 (1999); I. Bouchoule, M. Morinaga, C. Salomon, and D.S. Petrov, *ibid.* **65**, 033402 (2002).
[8] A.I. Safonov, S.A. Vasilyev, I.S. Yasnikov, I.I. Lukashevich, and S. Jaakkola, Phys. Rev. Lett. **81**, 4545 (1998).
[9] W. Hänsel, P. Hommelhoff, T.W. Hänsch, and J. Reichel, Nature (London) **413**, 498 (2001).
[10] H. Ott, J. Fortagh, G. Schlotterbeck, A. Grossmann, and C. Zimmermann, Phys. Rev. Lett. **87**, 230401 (2001).
[11] A. Griffin, W.-C. Wu, and S. Stringari, Phys. Rev. Lett. **78**, 1838 (1997).
[12] Yu. Kagan, E.L. Surkov, and G.V. Shlyapnikov, Phys. Rev. A **55**, R18 (1997).
[13] D. Guéry-Odelin, F. Zambelli, J. Dalibard, and S. Stringari, Phys. Rev. A **60**, 4851 (1999).
[14] U. Al Khawaja, C. Pethick, and H. Smith, J. Low Temp. Phys. **118**, 127 (2000); C. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, 1st ed. (Cambridge University Press, Cambridge, England, 2001).
[15] K. Huang, *Statistical Mechanics*, 2nd ed. (Wiley, New York, 1987).
[16] L.P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).
[17] H. Kreuzer, *Non Equilibrium Thermodynamics and its Statistical Foundations* (Clarendon, Oxford, 1981).
[18] Y. Castin, Session LXXII, in *Coherent Atomic Matter Waves*, "Les Houches," edited by R. Kaiser, C. Westbrook, and F. David (EDP Sciences, Springer-Verlag, Berlin, 2001).
[19] G. Baym and C.J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).
[20] Y. Castin and R. Dum, Phys. Rev. Lett. **77**, 5315 (1996).
[21] V.M. Perez-Garcia, H. Michinel, J.I. Cirac, M. Lewenstein, and P. Zoller, Phys. Rev. Lett. **77**, 5320 (1996).
[22] K.G. Singh and D.S. Rokhsar, Phys. Rev. Lett. **77**, 1667 (1996).
[23] M.J. Bijlsma and H.T.C. Stoof, Phys. Rev. A **60**, 3973 (1999).
[24] L. Boltzmann, in *Wissenschäftliche Abhandlungen*, edited by F. Hasenorl (J.A. Barth, Leipzig, 1909), Vol. II, p. 83.
[25] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
[26] L.P. Pitaevskii and A. Rosch, Phys. Rev. A **55**, R853 (1997).
[27] C. Menotti and S. Stringari, Phys. Rev. A (to be published), e-print cond-mat/0201158.
[28] D.F.V. James, Appl. Phys. B: Lasers Opt. **B66**, 181 (1998).
[29] T.K. Ghosh, Phys. Rev. A **63**, 013603 (2001).
[30] L.D. Landau, J. Phys. **5**, 71 (1941).
[31] D.S. Jin, M.R. Matthews, J.R. Ensher, C.E. Wieman, and E.A. Cornell, Phys. Rev. Lett. **78**, 764 (1997).
[32] D.M. Stamper-Kurn, H.-J. Miesner, S. Inouye, M.R. Andrews, and W. Ketterle, Phys. Rev. Lett. **81**, 500 (1998).
[33] F. Pereira Dos Santos, J. Léonard, J. Wang, C.J. Barrelet, F. Perales, E. Rasel, C.S. Unnikrishnan, M. Leduc, and C. Cohen-Tannoudji, Phys. Rev. Lett. **86**, 3459 (2001).