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Collective oscillations of a classical gas confined in harmonic traps

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Starting from the Boltzmann equation, we calculate the frequency and the damping of the collective oscillations of a classical gas confined by a harmonic potential. Both the monopole and quadrupole modes are considered in the presence of spherical as well as axially deformed traps. The relaxation time is calculated using a Gaussian ansatz which explicitly accounts for the occurence of quadrupole deformations in velocity space. Our approach provides an explicit description of the transition between the hydrodynamic and collisionless regimes. The predictions are in very good agreement with the results of a molecular-dynamics simulation carried out in a gas of hard spheres. [S1050-2947(99)02312-4]

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I. INTRODUCTION

After the experimental realization of Bose-Einstein condensation in trapped atomic gases [1], the investigation of collective excitations in these systems has become a very popular subject of research (see [2] for a recent theoretical review). At very low temperatures, when the the whole system is Bose-Einstein condensed, the motion is described by the hydrodynamic equations of superfluids. These equations, which can be directly derived starting from the mean-field Gross-Pitaevskii equation for the order parameter, give predictions [2,3] in very good agreement with experiments [4]. At higher temperatures the mean-field effects become less important, while collisional terms cannot be ignored. If the temperature is notably larger than the critical temperature for Bose-Einstein condensation the dynamical behavior of a dilute gas is well described by the Boltzmann equation. In this case two different regimes may occur: a collisional (hydrodynamic) regime characterized by conditions of local statistical equilibrium and a collisionless regime where the motion is described by the single-particle Hamiltonian. Differently from the case of uniform gases, also in the collisionless regime the system exhibits well-defined oscillations which are driven by the external confinement. The equations for the hydrodynamic regime were investigated in [5,6], while a phenomenological interpolation between the two regimes was proposed in [7]. A first attempt to describe the correction to the hydrodynamic limit using the Chapman-Enskog procedure was made in [8].

The purpose of this paper is to provide an analysis of the lowest oscillation modes (eigenfrequency and damping) in a harmonic trap with cylindrical symmetry, using an approximate solution of the classical Boltzmann equation. The main aim is to study the transition between the hydrodynamic and collisionless regimes. Our approach relies on a Gaussian ansatz for the distribution function. For harmonic trapping such an ansatz exactly reproduces the solution of the classical Boltzmann equation in both the hydrodynamic and collisionless regimes and is consequently expected to be a good approximation also in the intermediate regime. We thus perform a linear expansion of the collisional integral which leads to an analytic evaluation of the relaxation time for the quadrupole mode. The corresponding predictions are compared with the exact results of a numerical simulation based on molecular dynamics.

In our paper atoms behave like hard spheres, σ_0 being their total cross section which will be assumed to be energy independent. This is well satisfied in classical ultracold gases where collisions are completely characterized by the *s*-wave scattering length and the cross section is thus isotropic and in most cases energy independent.

II. METHOD OF AVERAGES

The starting point of our analysis is the Boltzmann equation for the phase space distribution function $f(\mathbf{r},\mathbf{v}_1,t)$ [10]:

$$\frac{\partial f}{\partial t}+\mathbf{v}_1\cdot\nabla_{\mathbf{r}}f+\frac{\mathbf{F}}{m}\cdot\nabla_{\mathbf{v}_1}f=I_{\rm coll}[f], \tag{1}$$

where

$$I_{\rm coll}[f]=\frac{\sigma_0}{4\pi}\int d^2\Omega\, d^3v_2|\mathbf{v}_2-\mathbf{v}_1|[f(\mathbf{v}_1')f(\mathbf{v}_2')-f(\mathbf{v}_1)f(\mathbf{v}_2)]$$

is the usual classical collisional integral. It accounts for elastic collisions between particles 1 and 2, with initial velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, and final velocities $\mathbf{v}_1'$ and $\mathbf{v}_2'$. The solid angle Ω gives the direction of the final relative velocity. The expression for the collisional term can be easily extended to include effects of both Bose and Fermi statistics [9]. Actually most of the results discussed in this paper hold also in the presence of quantum degeneracy, provided the system is not Bose-Einstein condensed and one can ignore mean-field effects. The quantitative estimates of collisional effects presented in this work will, however, be based on classical statistics.

The force $\mathbf{F}_{\rm trap}=-\nabla U_{\rm trap}(x,y,z)$ is produced by the confining potential which in the following will be chosen to be of harmonic form:

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$$U_{\rm trap}(x,y,z)=\frac{1}{2}m\omega_\perp^2(x^2+y^2)+\frac{1}{2}m\omega_z^2z^2. \tag{2}$$

We introduce the anisotropy parameter $\lambda=\omega_z/\omega_\perp$. For $\lambda=1$, one deals with an isotropic harmonic trap, for $\lambda^2\ll 1$ one has a cigar-shaped trap, and for $\lambda^2\gg 1$ a disk-shaped trap. Starting from Eq. (1), one can derive useful equations for the average of a general dynamical quantity $\chi(\mathbf{r},\mathbf{v})$:

$$\frac{d\langle\chi\rangle}{dt}-\langle\mathbf{v}\cdot\nabla_{\mathbf{r}}\chi\rangle-\left\langle\frac{\mathbf{F}_{\rm trap}}{m}\cdot\nabla_{\mathbf{v}}\chi\right\rangle=\langle\chi I_{\rm coll}\rangle, \tag{3}$$

where the average is taken in both position and velocity space:

$$\langle\chi\rangle=\frac{1}{N}\int d^3r\,d^3v\,f(\mathbf{r},\mathbf{v},t)\chi(\mathbf{r},\mathbf{v}). \tag{4}$$

As a consequence of the invariance properties of the cross section, the quantity $\langle\chi I_{\rm coll}\rangle$ can be written in the useful form

$$\langle\chi I_{\rm coll}\rangle=\frac{1}{4N}\int d^3r\,d^3v_1\,\Delta\chi I_{\rm coll}[f], \tag{5}$$

where $\Delta\chi=\chi_1+\chi_2-\chi_{1'}-\chi_{2'}$ with $\chi_i=\chi(\mathbf{r},\mathbf{v}_i)$. The collisional contribution (5) is equal to zero if χ corresponds to a dynamic quantity conserved during the elastic collision. This happens if χ can be written in the form [10,11]

$$\chi=a(\mathbf{r})+\mathbf{b}(\mathbf{r})\cdot\mathbf{v}+c(\mathbf{r})\mathbf{v}^2. \tag{6}$$

III. MONOPOLE OSCILLATION IN HARMONIC ISOTROPIC TRAPS

Let us consider a harmonic isotropic trapping potential ($\omega_x=\omega_y=\omega_z\equiv\omega_0$). As a first application of Eq. (3), one can immediately derive the behavior of the monopole mode [11–13] by computing the evolution of the square radius:

$$\frac{d\langle\mathbf{r}^2\rangle}{dt}=2\langle\mathbf{r}\cdot\mathbf{v}\rangle. \tag{7}$$

In order to obtain a closed set of equations one also needs the following equations:

$$\frac{d\langle\mathbf{r}\cdot\mathbf{v}\rangle}{dt}=\langle\mathbf{v}^2\rangle-\omega_0^2\langle\mathbf{r}^2\rangle \tag{8}$$

and

$$\frac{d\langle\mathbf{v}^2\rangle}{dt}=\omega_0^2\langle\mathbf{r}^2\rangle. \tag{9}$$

The collisional term does not contribute to the above equations because all the dynamic quantities satisfy the criterium (6). So there is no damping for the ''breathing'' mode of a classical dilute gas confined in a harmonic isotropic trap. The same is true if one includes quantum degeneracy effects in the collisional term.

By looking for solutions of Eqs. (7)–(9) evolving in time as $e^{i\omega t}$ one immediately finds the result $\omega=2\omega_0$, holding for all collisional regimes from the collisionless to the hydrodynamical one. The occurrence of this monopole undamped solution was pointed out by Boltzmann (see, for example, the discussion in [12]).

It is worth noticing that the frequency of the classical monopole oscillation in isotropic harmonic traps differs from the one of a Bose-Einstein condensed gas at $T=0$. In the latter case the monopole oscillation is still undamped, but the frequency, for large N, is $\omega=\sqrt{5}\omega_0$ [3]. The difference is the consequence of the combined effect of Bose-Einstein condensation and of the mean-field interaction. Furthermore, at finite temperature the monopole oscillation is expected to exhibit damping because of the coupling between the condensate and the thermal component of the gas.

Inclusion of mean-field effects on the left-hand side of the Boltzmann equation (1) would modify the structure of the system (7)–(9) of equations for the monopole oscillation which would no longer correspond to a closed set of equations. As a consequence, the monopole frequency would be shifted with respect to the value $2\omega_0$ and damped also in the classical regime.

IV. DAMPING OF THE QUADRUPOLE OSCILLATION

The purpose of this section is to investigate the quadrupole mode of a classical gas as well as its coupling with the monopole oscillation arising in anisotropic traps. In this case the solution of the Boltzmann equation exhibits damping and one has to deal explicitly with the collisional term. In the presence of anistropy, the $l_z=0$ component of the quadrupole is coupled with the monopole and one finally finds the following set of coupled equations:

$$\begin{aligned}
&\frac{d\langle\chi_1\rangle}{dt}-2\langle\chi_3\rangle=0,\\
&\frac{d\langle\chi_2\rangle}{dt}-2\langle\chi_4\rangle=0,\\
&\frac{d\langle\chi_3\rangle}{dt}-\langle\chi_5\rangle+\frac{2\omega_\perp^2+\omega_z^2}{3}\langle\chi_1\rangle+\frac{\omega_z^2-\omega_\perp^2}{3}\langle\chi_2\rangle=0,\\
&\frac{d\langle\chi_4\rangle}{dt}-\langle\chi_6\rangle+\frac{2\omega_z^2-2\omega_\perp^2}{3}\langle\chi_1\rangle+\frac{\omega_\perp^2+2\omega_z^2}{3}\langle\chi_2\rangle=0,\\
&\frac{d\langle\chi_5\rangle}{dt}+\frac{2\omega_z^2+4\omega_\perp^2}{3}\langle\chi_3\rangle+\frac{2\omega_z^2-2\omega_\perp^2}{3}\langle\chi_4\rangle=0,\\
&\frac{d\langle\chi_6\rangle}{dt}+\frac{4\omega_z^2-4\omega_\perp^2}{3}\langle\chi_3\rangle+\frac{4\omega_z^2+2\omega_\perp^2}{3}\langle\chi_4\rangle=\langle\chi_6 I_{\rm coll}\rangle,
\end{aligned} \tag{10}$$

where we have defined the quantities

$$\chi_1=\mathbf{r}^2,$$

$$\chi_2=2z^2-r_\perp^2,$$

$$\chi_3=\mathbf{r}\cdot\mathbf{v},$$

$$\chi_4=2zv_z-\mathbf{r}_\perp\cdot\mathbf{v}_\perp,$$

$$\chi_5 = \mathbf{v}^2,$$
$$\chi_6 = 2v_z^2 - v_\perp^2. \tag{11}$$

If the trap is isotropic ($\omega_\perp = \omega_z$), the set of Eqs. (10) decouples in two subsystems. One subsystem refers to the undamped monopole oscillations discussed in the preceding section. The other corresponds to the damped quadrupole mode. Notice that collisions affect only the last equation of Eq. (10). Actually, only the variable $\chi_6 = 2v_z^2 - v_\perp^2$ is not a conserved quantity and hence does not satisfy the criterium (6). The above results explicitly show that the relaxation mechanism of the oscillations described by Eqs. (10) is determined by the occurrence of quadrupole deformations in the velocity distribution which make the collisional integral $\langle \chi_6 I_{\rm coll}\rangle$ different from zero. In principle, this term should be calculated by a full solution of the Boltzmann equation, leading to an infinite hierarchy of equations.

The central point of our treatment is the approximate evaluation of $\langle \chi_6 I_{\rm coll}\rangle$ using a Gaussian ansatz of the form

$$f(\mathbf{r},\mathbf{v},t) = N\left(\frac{m}{2\pi}\right)^3 \frac{\alpha_\perp \alpha_z^{1/2}}{\theta_\perp \theta_z^{1/2}} e^{-mU_\perp^2/2\theta_\perp} \times e^{-mU_z^2/2\theta_z} e^{-m(\alpha_\perp r_\perp^2 + \alpha_z z^2)/2}, \tag{12}$$

where $r_\perp = \sqrt{x^2+y^2}$ and $\mathbf{U} = \mathbf{v} - \langle \mathbf{v}\rangle$. Equation (12) provides a natural generalization of the local equilibrium distribution, by introducing a deformation not only in coordinate space (taken into account by the α parameters), but also in velocity space. These deformations are of the quadrupole type and consequently are well suited to describe the relevant collisional effect entering the integral $\langle \chi_6 I_{\rm coll}\rangle$. Deformations of a similar form are responsible for the viscosity term in the Chapman-Enskog expansion of statistical mechanics [10]. One can show that, in the presence of harmonic trapping, the Gaussian ansatz (12) describes exactly the monopole and quadrupole oscillations both in the hydrodynamic and collisionless regimes. In the former case, the velocity distribution is isotropic and hence $\theta_\perp = \theta_z$. In the collisionless regime $\theta_\perp$ is instead different from θ_z, corresponding to configurations far from local equilibrium. Only in the presence of isotropic trapping and for the monopole oscillation do the hydrodynamic and collisionless solutions coincide. In this case the ansatz (12), with $\theta_\perp = \theta_z$ and $\alpha_\perp = \alpha_z$ provides an exact solution of the Boltzmann equation.

In the limit of small oscillations around the equilibrium configuration, the axial and transverse temperatures can be expanded around the equilibrium value θ_0:

$$\theta_\perp = \theta_0 + \delta\theta_\perp,$$
$$\theta_z = \theta_0 + \delta\theta_z, \tag{13}$$

and one finds

$$\langle \chi_6 \rangle = \frac{2}{m}(\delta\theta_z - \delta\theta_\perp) \tag{14}$$

showing that $\langle \chi_6\rangle$ is directly sensitive to the anisotropy of the velocity distribution. The collisional contribution to the equation for χ_6 [last equation of the system (10)], can be also expressed in terms of this anisotropy. By inserting Eq. (12) into Eq. (5) with $\chi = \chi_6$ we obtain, after linearization, the expression (see the Appendix)

$$\langle \chi_6 I_{\rm coll}\rangle = -\frac{2}{\tau}\frac{(\delta\theta_z - \delta\theta_\perp)}{m} = -\frac{\langle \chi_6\rangle}{\tau}, \tag{15}$$

where the relaxation time τ is given by

$$\tau = \frac{5}{4\gamma_{\rm coll}}. \tag{16}$$

Equation (16) provides an explicit link between the classical collision rate [16]

$$\gamma_{\rm coll} = \frac{n(0)v_{\rm th}\sigma_0}{2} \tag{17}$$

giving the number of collisions undergone by a given atom per unit of time, and the relaxation time for the quadrupole mode. In Eq. (17) $v_{\rm th} = \sqrt{8\theta_0/\pi m}$ is the thermal velocity and $n(0)$ is the central density. Notice that this relationship is predicted to be independent of the anisotropy λ of the trap since the spatial dependence of the distribution function (12) can be factorized in the calculation of the collisional integral.

Result (15), when inserted into Eqs. (10), permits us to obtain a linear and closed set of equations which can be solved by looking for solutions of the type $e^{i\omega t}$. The associated determinant then yields the dispersion law

$$(\omega^2 - 4\omega_z^2)(\omega^2 - 4\omega_\perp^2) - \frac{i}{\omega\tau}\left(\omega^4 - \frac{2}{3}\omega^2(5\omega_\perp^2 + 4\omega_z^2) + 8\omega_\perp^2\omega_z^2\right) = 0. \tag{18}$$

The first term of Eq. (18) corresponds to the dispersion law for the pure collisionless regime ($\omega\tau \to \infty$). In this case the eigenfrequencies coincide with the ones predicted by the single-particle harmonic-oscillator Hamiltonian: $\omega_{\rm CL} = 2\omega_z$ and $2\omega_\perp$. Viceversa, the term multiplying $1/\tau$ refers to the pure hydrodynamical regime ($\omega\tau \to 0$). For a spherical trap, one gets $\omega_{\rm HD} = \sqrt{2}\omega_0$ and $2\omega_0$ for the quadrupole and monopole modes, respectively. For a cigar-shaped configuration ($\lambda^2 \ll 1$) the two hydrodynamic solutions have instead the form $\omega_{\rm HD} = \sqrt{12/5}\omega_z$ and $\sqrt{10/3}\omega_\perp$ [5], while for a disk trap ($\lambda^2 \gg 1$), one finds $\omega_{\rm HD} = \sqrt{8/3}\omega_z$ and $\sqrt{3}\omega_\perp$.

Formula (18), which provides the proper interpolation between the collisionless and hydrodynamic regimes, can be simplified in the case of a spherical, cigar, and disk-shaped trap. In fact, the dispersion law (18) can be written in all these limiting cases in the useful form

$$\omega^2 = \omega_{\rm CL}^2 + \frac{\omega_{\rm HD}^2 - \omega_{\rm CL}^2}{1 + i\omega\tilde{\tau}} \tag{19}$$

typical of relaxation phenomena [7,14]. The time $\tilde{\tau}$ is related to τ by a simple numerical factor. For example, $\tilde{\tau} = \tau$ for the quadrupole mode in the spherical case, and $\tilde{\tau} = 6\tau/5$ ($\tilde{\tau} = 4\tau/3$) for the lowest mode of the cigar- (disk) shaped configuration. A relevant feature of Eq. (19) is the presence of an imaginary part, associated with the damping of the oscillation. By writing the frequency as $\omega=\omega_r+i\Gamma$ one finds, assuming $\Gamma\ll\omega_r$,

$$\Gamma\simeq\frac{\tilde{\tau}}{2}\frac{\omega_{\rm CL}^2-\omega_{\rm HD}^2}{1+(\omega_r\tilde{\tau})^2}. \tag{20}$$

Notice that the damping depends crucially on the difference between the frequencies calculated in the collisionless and hydrodynamic regimes and exactly vanishes when these frequencies coincide. This happens, for example, in the monopole case for isotropic trapping, as discussed in the previous section [15].

In the hydrodynamic limit $(\omega_r\tilde{\tau}\ll 1)$ the damping predicted by Eq. (20) takes the form

$$\Gamma_{\rm HD}\simeq\frac{\tilde{\tau}}{2}(\omega_{\rm CL}^2-\omega_{\rm HD}^2), \tag{21}$$

while in the opposite regime $(\omega_r\tilde{\tau}\gg 1)$ one gets

$$\Gamma_{\rm CL}\simeq\frac{1}{2\omega_{\rm CL}^2\tilde{\tau}}(\omega_{\rm CL}^2-\omega_{\rm HD}^2). \tag{22}$$

A maximum for Γ is found at $\omega_r\tilde{\tau}\sim 1$, leading to $\Gamma\sim(\omega_{\rm CL}^2-\omega_{\rm HD}^2)/\omega_r$. Around this value, the approximation leading to Eq. (20) is no longer accurate, and one should rather use Eq. (18) or Eq. (19).

In a similar way one can also investigate the frequency and the damping of the $l_z=2$ quadrupole modes for a harmonic trap with cylindrical symmetry. In this case, the dispersion law can be exactly written in the form (19) with $\tilde{\tau}=\tau$, $\omega_{\rm CL}=2\omega_\perp$, and $\omega_{\rm HD}=\sqrt{2}\omega_\perp$.

V. NUMERICAL SIMULATION

In this section, we present results for the dispersion law arising from a numerical simulation, based on molecular dynamics. Our aim is to check the quality of prediction (16) for the relaxation time given by the Gaussian approximation discussed in the preceding section. We consider $N=2\times10^4$ particles moving in the potential (2). Binary elastic collisions are taken into account using a boxing technique [17,18]. At each time step δt, the position of each particle is discretized on a square lattice with a step ξ. The volume ξ^3 of a box is chosen such that the average occupation $p_{\rm occ}$ of any box is much smaller than 1. Collisions occur only between two particles occupying the same box, and the time step δt is adjusted in such a way that the probability $p_{\rm coll}$ of a collisional event during δt is also much smaller than 1. We choose typically $p_{\rm occ}\sim p_{\rm coll}\sim 5\%$.

Initial conditions for exciting the lowest energetical mode are obtained by a deformation in coordinate and velocity spaces of the cloud along the weak axes, keeping the phase-space density constant. We have checked that this method leads to the excitation of only the lowest frequency mode. Then, we let the cloud evolve. The damped oscillation of the variable $\chi_2=2z^2-r_\perp^2$ is analyzed for different choices of the collision rate. As an example, the oscillation frequency and the damping for a cigar-shaped trap ($\lambda=1/10$) are plotted (solid circles), respectively, on Figs. 1 and 2. One observes that the frequency decreases as the ratio $\omega_z/\gamma_{\rm coll}$ decreases and tends asymptotically to the hydrodynamic value $\sqrt{12/5}\omega_z=1.55\omega_z$. For large value of $\omega_z/\gamma_{\rm coll}$, the frequency instead approaches the collisionless value $2\omega_z$. By performing a least-squares fit with formula (18), we obtain

$$\tau=(1.28\pm0.05)\frac{1}{\gamma_{\rm coll}}, \tag{23}$$

which agrees well with the Gaussian prediction (16). The full line on Figs. 1 and 2 corresponds to Eq. (18) with τ given by

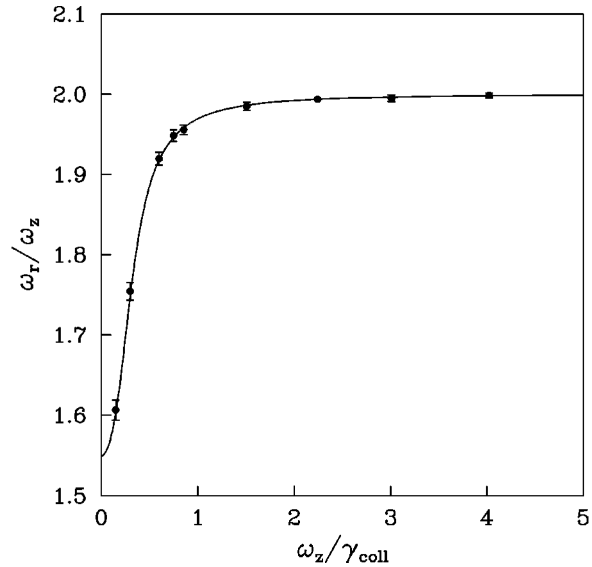

FIG. 1. Real part of the frequency of the $l_z=0$ mode of a classical gas confined in a cigar-shaped trap ($\lambda=1/10$), versus $\omega_z/\gamma_{\rm coll}$. The solid curve represents the prediction of the Gaussian ansatz. The circles are the numerical results obtained with a molecular dynamics simulation.

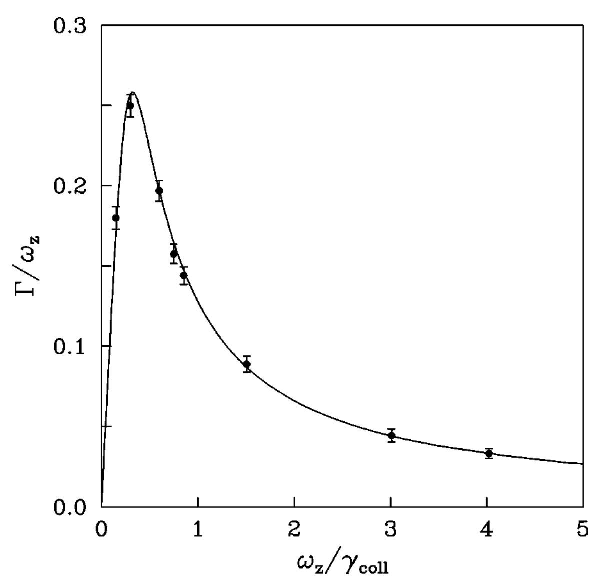

FIG. 2. Damping of the $l_z=0$ mode of a classical gas confined in a cigar-shaped trap ($\lambda=1/10$), versus $\omega_z/\gamma_{\rm coll}$. The notations for the line and the markers are the same as in Fig. 1.

the Gaussian ansatz prediction (16). We have also checked that result (23) is independent of the value of λ, consistently with the prediction of the Gaussian ansatz.

VI. CONCLUSION

In this work, we have presented an investigation of the collective frequencies of a classical gas trapped in a harmonic potential well. Starting from the classical Boltzmann equation, we have derived a set of coupled equations for the averages of the relevant dynamic variables associated with the monopole and quadrupole modes. The relaxation time for the quadrupole oscillation was evaluated by a Gaussian ansatz for the distribution function and the quality of the approximation was checked by a numerical simulation based on molecular dynamics.

The results of the present work suggest that the Gaussian ansatz is very accurate for investigating the damping of the quadrupole oscillation and in general the transition between the hydrodynamic and the collisionless regimes of this mode. Our approach is based on the use of the classical Boltzmann equation as a natural starting point and consequently neglects the possible occurence of mean-field interactions. This effect should be responsible, in particular, for the occurrence of damping in the monopole oscillation also in the case of isotropic harmonic trapping. The investigation of the mean-field corrections as well as the inclusion of quantum statistical effects in the collisional term will be the object of a future investigation.

APPENDIX: COLLISIONAL INTEGRAL

The Appendix is devoted to the explicit calculation of the collisional integral $\langle \chi_6 I_{\rm coll}\rangle$. After linearization with respect to $\delta\theta_\perp$ and $\delta\theta_z$, this integral reads

$$\begin{aligned}\langle \chi_6 I_{\rm coll}\rangle = \frac{1}{32\pi}\frac{m\sigma_0}{N\theta_0^2}\int d^3r d^3U_1 d^3U_2 \\ \times|\mathbf{U}_1-\mathbf{U}_2| d^2\Omega f_0(1) f_0(2)\Delta\chi_6[\delta\theta_\perp((U_\perp^2)_{1'} \\ +(U_\perp^2)_{2'}-(U_\perp^2)_1-(U_\perp^2)_2)+\delta\theta_z((U_z^2)_{1'} \\ +(U_z^2)_{2'}-(U_z^2)_1-(U_z^2)_2)],\end{aligned} \tag{A1}$$

where f_0 is the Gaussian (12) evaluated at equilibium, and $\Delta\chi_6=(\chi_6)_1+(\chi_6)_2-(\chi_6)_{1'}-(\chi_6)_{2'}$. Let us introduce the center-of-mass velocity $\mathbf{C}$ and the relative velocity before $(\mathbf{V})$ and after $(\mathbf{V}')$ collision:

$$\begin{aligned}\mathbf{U}_1=\mathbf{C}+\mathbf{V}/2, \\ \mathbf{U}_2=\mathbf{C}-\mathbf{V}/2, \\ \mathbf{U}_{1'}=\mathbf{C}+\mathbf{V}'/2, \\ \mathbf{U}_{2'}=\mathbf{C}-\mathbf{V}'/2.\end{aligned} \tag{A2}$$

The conservation of kinetic energy during an elastic collision ensures

$$V^2=V'^2, \tag{A3}$$

so that the collisional integral can be rewritten in the form

$$\begin{aligned}\langle \chi_6 I_{\rm coll}\rangle = -(\delta\theta_z-\delta\theta_\perp)\frac{3}{128\pi}\frac{m\sigma_0}{N\theta_0^2} \\ \times\int d^3r dV d^3C d^2\Omega d^2\Omega_V \\ \times V^3 f_0(1) f_0(2)[V_z^2-V_{z'}^2]^2,\end{aligned} \tag{A4}$$

where Ω denotes the solid angle between $\mathbf{V}$ and $\mathbf{V}'$, and Ω_V fixes the absolute angle of $\mathbf{V}$. Let us first calculate the angular integral:

$$I_\Omega \equiv \int d^2\Omega_V d^2\Omega [V_z^2-V'^2_z]^2. \tag{A5}$$

As the integration is made on all relative velocity angles, one can perform a change of variables and integrate over the angles Ω_V and $\Omega_{V'}$ independently. Using spherical coordinates, one then easily finds the result

$$I_\Omega = V^4\frac{128\pi^2}{45}. \tag{A6}$$

The calculation of the collisional integral (A4) is now straightforward as it involves only Gaussian integrals. We finally obtain the useful expression

$$\langle \chi_6 I_{\rm coll}\rangle = -(\delta\theta_z-\delta\theta_\perp)\frac{4}{5m}v_{\rm th}\sigma_0 n(0) \tag{A7}$$

for the collisional integral, where $v_{\rm th}$ is the thermal velocity. Result (A7) permits us to derive the main equations (15)–(17) used in Sec. IV to calculate the relaxation time τ and the damping of the quadrupole mode.

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