Classical phase-space approach for coherent matter waves

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We investigate a classical phase-space approach of matter-wave propagation based on the truncated Wigner equation (TWE). We show that such a description is suitable for ideal matter waves in quadratic time-dependent confinement as well as for harmonically trapped Bose-Einstein condensates in the Thomas-Fermi regime. In arbitrary interacting regimes, the TWE combined with the moment method yields the low-energy spectrum of a condensate as predicted by independent variational methods. TWE also gives the right breathing-mode frequency for long-ranged interactions decaying as $1/r^2$ in three dimensions and for a contact potential in two dimensions. Quantum signatures, beyond the TWE, may only be found in the condensate dynamics beyond the regimes of classical phase-space propagation identified here.

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Since being introduced in the early days of quantum mechanics, Wigner distributions have been successfully applied in many areas of physics. In optics, they are used to describe partially coherent light beams [1], and, in atomic physics, they are used to characterize atom interferometers [2] or Bose-Einstein condensates (BECs) [3]. The Wigner distribution obeys a propagation equation involving a series of differential operators weighted by increasing powers of $\hbar$ [4], which can be identified as the quantum terms of the Wigner equation. The classical limit of this equation, obtained by taking the limit $\hbar \to 0$, is referred to as the truncated Wigner equation (TWE). This equation has been used with stochastic classical fields to study quantum gases [5]. Here, we propose to use it instead with the modes of a coherent atomic beam. When combined with the moment method, this reveals an accurate yet simple way to capture the low-energy dynamics of a BEC. It is, for instance, remarkable that the TWE correctly describes the time-of-flight expansion of condensates in the limit of vanishing interactions as well as in the Thomas-Fermi limit. Our treatment, which studies the agreement between the TWE and the full Wigner equation, also identifies the circumstances under which quantum signatures (i.e., effects due to the quantum terms of the Wigner equation) may appear in the dynamics of BECs.

After a quick reminder about the general Wigner equation, we study its applications for the propagation of ideal matter waves in time-dependent quadratic potentials. We retrieve the $ABCD$ propagation formalism [6] for coherent guided atom optics [7], suitable for investigating atomic beams, which propagate in time domain [8] through a variety of atomic-optical elements. It can also be used to study the transverse stability [9] of an atomic resonator [10], to define the quality factor of an atomic beam [11], or to investigate the generation of atomic-optical caustics [11] caused by a sudden potential change. We also analyze the predictions of the TWE for the expansion and for the low-energy spectrum of interacting atomic waves, which evolve in harmonic traps. The consistency of the TWE with a universal prediction for long-ranged $1/r^2$ interactions is verified.

We recall the form of the Wigner equation in the presence of a general two-body potential $V(\mathbf{r})$. The nonrelativistic atomic-field operator $\hat{\psi}$ obeys the general propagation equation,

$$i\hbar\frac{\partial\hat{\psi}}{\partial t}(\mathbf{r},t) = -\frac{\hbar^2}{2m}\Delta\hat{\psi}(\mathbf{r},t) + U_0(\mathbf{r},t)\hat{\psi}(\mathbf{r},t) + \int d\mathbf{r}'\hat{\psi}^\dagger(\mathbf{r}',t)V(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}',t)\hat{\psi}(\mathbf{r},t), \quad (1)$$

where $U_0(\mathbf{r},t)$ is the time-dependent external potential experienced by the atoms. We assume the sample to be well described by the mean-field approximation and in a number state associated with a macroscopic single mode ϕ. The associated Wigner distribution is

$$W(\mathbf{r},\mathbf{p},t) = \int \frac{d\mathbf{r}'}{(2\pi\hbar)^3}\phi\left(\mathbf{r}+\frac{\mathbf{r}'}{2},t\right)\phi^*\left(\mathbf{r}-\frac{\mathbf{r}'}{2},t\right)e^{(-i/\hbar)\mathbf{p}\cdot\mathbf{r}'}. \quad (2)$$

The mode ϕ satisfies a Schrödinger equation with an effective Hamiltonian $H(\mathbf{r},\mathbf{p},t) = \mathbf{p}^2/2m + U(\mathbf{r},t)$, where $U(\mathbf{r},t) = U_0(\mathbf{r},t) + U^{\mathrm{mf}}(\mathbf{r},t)$ and $U^{\mathrm{mf}}(\mathbf{r},t) = \int d\mathbf{r}'|\phi(\mathbf{r}',t)|^2V(\mathbf{r}-\mathbf{r}')$. For a contact-interaction potential $V(\mathbf{r}) = g\delta(\mathbf{r})$, this Schrödinger equation reduces to the usual time-dependent Gross-Pitaevskii equation. The Wigner distribution associated with a wave function governed by a Hamiltonian $H(\mathbf{r},\mathbf{p},t)$ satisfies the transport equation [12,13] $\partial W/\partial t = -i\mathcal{L}[W]$, where $\mathcal{L}$ is the Liouvillian operator,

$$\mathcal{L} = H(\mathbf{r},\mathbf{p},t)\left[\frac{2i}{\hbar}\sin\frac{\hbar}{2}\overleftrightarrow{\Lambda}\right] \quad \text{with}$$
$$\overleftrightarrow{\Lambda} = \sum_{\eta=1,2,3}\frac{\overleftarrow{\partial}}{\partial r_\eta}\frac{\overrightarrow{\partial}}{\partial p_\eta} - \frac{\overleftarrow{\partial}}{\partial p_\eta}\frac{\overrightarrow{\partial}}{\partial r_\eta}. \quad (3)$$

We have identified the vectors $\mathbf{r} = (x,y,z)$ and $\mathbf{p} = (p_x,p_y,p_z)$ with (r_1,r_2,r_3) and (p_1,p_2,p_3). The differentiation operators $\overrightarrow{\partial}$ and $\overleftarrow{\partial}$ act on the Wigner distribution $W(\mathbf{r},\mathbf{p},t)$ and on the Hamiltonian $H(\mathbf{r},\mathbf{p},t)$, respectively. The Wigner equation can be recast in an explicit differential series ordered by increasing powers of the constant $\hbar$,

$$W_t = \sum_{i=1}^{3}\left(-\frac{p_i}{m}W_{r_i} + U_{r_i}W_{p_i}\right) + \sum_{\substack{n=1\\ n_1+n_2+n_3=2n+1}}^{+\infty} \frac{(i\hbar)^{2n}}{2^{2n}n_1!n_2!n_3!}U_{r_1^{n_1}r_2^{n_2}r_3^{n_3}}W_{p_1^{n_1}p_2^{n_2}p_3^{n_3}}. \quad (4)$$

To alleviate notations, from now on, we note the partial differentiation with respect to time, space, and momentum coordinates thanks to an index (i.e., $W_{p_1^{n_1}p_2^{n_2}p_3^{n_3}}$ is the derivative of the Wigner distribution $\frac{\partial^{n_1}}{\partial p_1^{n_1}}\frac{\partial^{n_2}}{\partial p_2^{n_2}}\frac{\partial^{n_3}}{\partial p_3^{n_3}}W$). The TWE is obtained by taking the classical limit $\hbar \to 0$ [i.e., by discarding the last term of the right-hand side (rhs) in Eq. (4)].

Let us first treat the propagation of ideal atomic waves in time-dependent quadratic potentials. In this regime, the full Wigner equation exhibits no quantum terms and, thus, reduces to the TWE. We show that the Wigner equation, or, here, equivalently the TWE, can be used to exactly recover the $ABCD$ propagation formalism. This phase-space method had already been verified by direct integration for one-dimensional Laguerre-Gaussian modes [14], and analogously for partially coherent matter waves [15]. The Wigner equation offers a compact proof of this result, by avoiding the computation of propagation integrals.

Consider a general quadratic Hamiltonian (without atomic interactions) expressed with the conventions of [6]

$$H = \frac{{}^T\mathbf{p}\cdot\beta\cdot\mathbf{p}}{2m} - {}^T\mathbf{r}\alpha\mathbf{p} - \frac{m}{2}{}^T\mathbf{r}\gamma\mathbf{r} - m\,\vec{g}\cdot\mathbf{r} + \vec{f}\cdot\mathbf{p}, \quad (5)$$

where α, β, and γ are 3×3 time-dependent real matrices satisfying ${}^T\alpha = -\alpha$, ${}^T\beta = \beta$, ${}^T\gamma = \gamma$, and $\vec{f}$ and $\vec{g}$ are time-dependent vectors. ^{T} stands for the matrix or the vector transposition. One readily sees that the full Wigner equation is, here, simply given by its classical part (TWE) $y_t = -H_{\mathbf{p}}(\mathbf{r},\mathbf{p})y_{\mathbf{r}} + H_{\mathbf{r}}(\mathbf{r},\mathbf{p})y_{\mathbf{p}}$. We introduce the phase-space vector $\mathbf{R}(t) = (\mathbf{r},\mathbf{p}/m)$. We expect the following phase-space map [6]:

$$\mathbf{R}(t) = M(t,t_0)\mathbf{R}(t_0) + \mathbf{R}_0(t,t_0). \quad (6)$$

$\mathbf{R}_0(t,t_0) = (\vec{\xi},\vec{\phi})$ is a source term to be determined later. $M(t,t_0)$ is a 6×6 matrix, referred to as the $ABCD$ matrix associated with the evolution between the initial time t_0 and the final time t: $M(t,t_0) = [A|B,C|D]$. We used the matrix notation $M = [m_{11}|m_{12},m_{21}|m_{22}]$—$m_{ij}$ as the element of the ith row and jth column—and A,B,C,D are 3×3 matrices, which depend on the pair of instants (t,t_0). The $ABCD$ matrix satisfies the symplectic relation $M^{-1} = [{}^TD|-{}^TB,{}^TC|{}^TA]$. The phase-space map of Eq. (6) and the symplectic relation lead us to consider the following ansatz as a possible solution to the TWE:

$$F(\mathbf{r},\mathbf{p},t) = W\Big({}^TD(\mathbf{r}-\vec{\xi}) - \frac{1}{m}{}^TB(\mathbf{p}-m\,\vec{\phi}), -m\,{}^TC(\mathbf{r}-\vec{\xi}) + {}^TA(\mathbf{p}-m\,\vec{\phi}),t_0\Big). \quad (7)$$

This ansatz indeed satisfies the Wigner equation if the following relation is fulfilled for any phase-space point:

$$T_1\mathbf{r}\cdot W^0_{\mathbf{r}} + T_2\frac{\mathbf{p}}{m}\cdot W^0_{\mathbf{r}} + mT_3\mathbf{r}\cdot W^0_{\mathbf{p}} + T_4\mathbf{p}\cdot W^0_{\mathbf{p}} + \vec{T}_5\cdot W^0_{\mathbf{r}} + m\vec{T}_6\cdot W^0_{\mathbf{p}} = 0. \quad (8a)$$

with

$$T_1 = {}^T\dot{D} - {}^TD^T\alpha - {}^TB\gamma, \quad T_2 = -{}^T\dot{B} + {}^TD^T\beta - {}^TB\alpha$$
$$T_3 = -{}^T\dot{C} + {}^TC^T\alpha + {}^TA\gamma, \quad T_4 = {}^T\dot{A} - {}^TC\beta - {}^TA^T\alpha$$
$$\vec{T}_5 = {}^T\dot{B}\vec{\phi} + {}^TB\dot{\vec{\phi}} - {}^TB\vec{g} - ({}^T\dot{D}\vec{\xi} - {}^TD\dot{\vec{\xi}} + {}^TD\vec{f})$$
$$\vec{T}_6 = {}^T\dot{C}\vec{\xi} + {}^TC\dot{\vec{\xi}} - {}^TC\vec{f} - ({}^T\dot{A}\vec{\phi} + {}^TA\dot{\vec{\phi}} - {}^TA\vec{g}) \quad (8b)$$

The upper dots stand for time derivatives, and the lower dots denote scalar products. The terms $W^0_{\mathbf{r}}, W^0_{\mathbf{p}}$ correspond to the gradients of the Wigner distributions toward the position and momentum vectors, respectively, evaluated at the initial time t_0. It is now convenient to introduce the matrix N and the vector S defined from the Hamiltonian of Eq. (5) as $N = [\alpha|\beta,\gamma|\alpha]$ and $\mathbf{S} = (\vec{f},\vec{g})$. Equation (8) implies that $T_{1,2,3,4} = 0_3$ and $\vec{T}_{5,6} = \vec{0}$, i.e. the parameters $A,B,C,D,\vec{\xi}$, and $\vec{\phi}$ satisfy the differential equations

$$\frac{dM}{dt} = NM, \qquad \frac{d}{dt}(M^{-1}R_0) = M^{-1}S, \quad (9)$$

and satisfy the initial conditions $\vec{\xi}(t_0,t_0) = \vec{\phi}(t_0,t_0) = \vec{0}$, $A(t_0,t_0) = D(t_0,t_0) = I_3$, and $B(t_0,t_0) = C(t_0,t_0) = 0_3$. I_3 and 0_3 denote, respectively, the identity and the zero 3×3 matrix. The system obtained in Ref. [6] for the parameters $A,B,C,D,\vec{\xi}$, and $\vec{\phi}$ is equivalent to Eq. (9). With the adequate initial values for $A,B,C,D,\vec{\xi}$, and $\vec{\phi}$, the ansatz F coincides initially with the Wigner distribution and satisfies the same differential equation: The Wigner function, thus, evolves according to Eqs. (7) and (9).

The TWE also gives the correct predictions for the expansion of a dilute BEC trapped in the Thomas-Fermi limit, in particular, the correct expansion law [16] expected in a time-of-flight experiment and the corresponding low-lying mode frequencies [17]. Here, we summarize only the main points of the argument. The BEC, subject to contact interactions, evolves in a time-dependent quadratic external potential $U(\mathbf{r},t) = \sum_{i=1}^{3} m\omega_i^2(t)r_i^2/2$. Our approach combines the technique of scaling factors already exploited by several authors [16,18,19] with the TWE. The transposition of scaling laws to Wigner functions consists of searching for an ansatz of the form $W(\mathbf{r},\mathbf{p},t) \simeq \widetilde{W}(\mathbf{r}',\mathbf{p}',0)$ with a gauge transform [20] $(r_i',p_i') = (r_i/b_i(t), b_i(t)p_i - m\dot{b}_i(t)r_i)$. This ansatz depends on time only through the scaling parameters b_i. Its adequacy with the exact Wigner distribution in the Thomas-Fermi limit is not obvious and has not been considered so far to our knowledge. For this purpose, one writes the Wigner distribution as $W(\mathbf{r},\mathbf{p},t) = \widetilde{W}(\mathbf{r}',\mathbf{p}',0) + \delta\widetilde{W}$, and shows that the time-dependent correction $\delta\widetilde{W}$ vanishes in this regime. By inserting this expression in the TWE, one sees that this correction is driven by a term, which vanishes if the scaling parameters satisfy $\ddot{b}_i = \omega_i^2(0)/(b_i\prod_{j=1}^{3}b_j) - b_i\omega_i^2(t)$.

By using a procedure suggested in Ref. [16], one can show that δW remains negligible in the Thomas-Fermi limit.

A powerful theoretical technique to track the phase-space motion is provided by the use of moments, and has been extensively used in combination with nonlinear Schrödinger-like equations, in optics and electromagnetism [21,22]. It also turns out to be useful for deriving low-lying collective modes of a harmonically trapped Bose gas degenerate or not. In the following, we detail the transposition of this technique within the Wigner formalism. Moments are defined with the Wigner distribution by $\langle a(\mathbf{r},\mathbf{p})\rangle(t) = \int d^3\mathbf{r}d^3\mathbf{p}a(\mathbf{r},\mathbf{p})W(\mathbf{r},\mathbf{p},t)$ with $a(\mathbf{r},\mathbf{p})$ as a generic monomial in the position and momentum coordinates. By convention, we define the order of the moment as the total power of the monomial. The orders in position and momentum are the partial powers associated with the product of the position and the momentum coordinates, respectively, inside the moment.

It is insightful to investigate whether the predictions of the TWE, which regard the dynamics of moments, differ from those of the full Wigner equation. From the full Wigner equation, Eq. (4), we deduce that the first-order time derivative of moments with an order in momentum strictly less than 3 can be expressed exactly by considering only the classical terms of this equation, that is, the TWE. Indeed, by considering the moment $\langle r_1^{a_1} r_2^{a_2} r_3^{a_3} p_1^{b_1} p_2^{b_2} p_3^{b_3}\rangle$, one sees that its time derivative involves only the terms of the quantum series of Eq. (4), which satisfy $n_k \leqslant b_k$ for $k = 1,2,3$. In particular, the first-order time derivative of any second-order moment is given exactly by the TWE.

For ideal matter waves propagating in quadratic potentials, the two terms in the rhs of the TWE, namely, $-p_i/mW_{r_i}$ and $U_{r_i}W_{p_i}$, couple the motion of a moment only to other moments of equal or lower order. With the previous remark about the time derivative of moments, one can see that the TWE exactly describes the dynamics of the second-order moments. In addition, any moment also follows a closed set of equations of motion. In contrast, if the effective potential U contains polynomials beyond the second order—which come either from the external potential or from the mean-field interactions—these properties are no longer verified. Indeed, the second classical term in the rhs, $U_{r_i}W_{p_i}$, couples the motion of any moment with a nonzero momentum order to higher-order moments. Second-order moments are then coupled to a hierarchy of moments of arbitrary high order, some of which are influenced by the quantum terms of the Wigner equation. In this case, one must use the full Wigner equation to capture the moment dynamics.

By applying the TWE to specific moments, which are not directly influenced by the quantum terms, it is still possible to exactly infer the low-energy mode frequencies of an interacting sample in the presence of either contact, dipolar, or $1/r^2$ interactions. Now, we focus on the motion of the second-order moments. Their first-order time derivative can be exactly evaluated without considering the quantum terms in the Wigner equation. In this sense, the equations of motion [23],

$$\frac{d\langle r_i^2\rangle}{dt} = \frac{2}{m}\langle r_i p_i\rangle, \quad \frac{d\langle r_i p_i\rangle}{dt} = \frac{1}{m}\langle p_i^2\rangle - m\omega_i^2\langle r_i^2\rangle - \langle r_i U_{r_i}^{\mathrm{mf}}\rangle \tag{10}$$

are classical and can be regarded as TWE predictions.

Let us prove that the TWE respects the universal ground-frequency invariance [24] expected when one adds an interaction potential V, which satisfies the scaling law $V(\lambda\mathbf{r}) = \lambda^{-2}V(\mathbf{r})$. In a three-dimensional system, this property is only fulfilled by the long-ranged $1/r^2$ potential, but, in a two-dimensional system (such as a condensate with a frozen external degree of freedom), it is also valid for contact potentials such as $g\delta(\mathbf{r})$. Similar to the derivation of the virial theorem in mechanics, one obtains, by differentiation of Eqs. (10):

$$\frac{m}{2}\frac{d^2\langle r^2\rangle}{dt^2} = \frac{\langle p^2\rangle}{m} - m\omega_0^2\langle r^2\rangle + \langle \mathbf{r}\cdot U_{\mathbf{r}}^{\mathrm{mf}}\rangle. \tag{11}$$

The scaling law of the interaction potential implies that $\langle \mathbf{r}\cdot U_{\mathbf{r}}^{\mathrm{mf}}\rangle = -2\langle U^{\mathrm{mf}}\rangle$. One obtains the closed equation $d^2\langle r^2\rangle/dt^2 + 4\omega_0^2\langle r^2\rangle = 4E/m$ with the total energy $E = \langle p^2\rangle/2m + m\omega_0^2\langle r^2\rangle/2 + \langle U^{\mathrm{mf}}\rangle$. The lowest-mode frequency $2\omega_0$ is, thus, immune to the addition of an interacting potential with the earlier scaling property. We now show that the classical Eqs. (10) are sufficient to predict the low-energy modes of an interacting condensate. We choose an interaction potential in the general form [25] $V(\mathbf{r}) = g_c\delta(\mathbf{r}) + g_d(1 - 3\cos^2\theta_r)/r^2$. It contains a contact term and a dipolar contribution associated with a permanent moment; dipoles are aligned along the vector $\mathbf{e}_3$ and $\cos\theta_r = (\mathbf{r}\cdot\mathbf{e}_3)/r$. As such, the TWE cannot be solved analytically, but we can investigate its predictions thanks to a Gaussian ansatz,

$$W(\mathbf{r},\mathbf{p},t) = \frac{1}{(\pi\hbar)^3}\prod_{i=1,2,3}\exp\left[-\frac{(r_i - r_{0i})^2}{\lambda_i^2}\right] \times \exp - \left[\frac{\lambda_i^2}{\hbar^2}(p_i - \hbar\alpha_i - 2\hbar\beta_i r_i)^2\right]. \tag{12}$$

r_{0i} is the average position, λ_i is the width of the wave packet, and the parameters α_i,β_i define a quadratic wave front. By using the relation $d\langle r_i\rangle/dt = \langle p_i\rangle/m$ and the first Eq. (10), one finds $\alpha_i = (m/\hbar)[\dot r_{0i} - (\dot\lambda_i/\lambda_i)r_{0i}]$ and $\beta_i = m\dot\lambda_i/(2\hbar\lambda_i)$. By using these expressions in Eq. (12) yields the scaling gauge transform introduced previously. For each coordinate, Eqs. (10) give an equation of motion similar to Eq. (11). The second-order moments in position and in momentum can be written with the ansatz parameters as $\langle r_i^2\rangle = r_{0i}^2 + \lambda_i^2/2$ and $\langle p_i^2\rangle = m^2(\dot r_{0i}^2 + \dot\lambda_i^2/2) + \hbar^2/(2\lambda_i^2)$. By reporting these expressions in the equations of motion, and by using $\ddot r_{0i} = -\omega_i^2 r_{0i}$, one finds that the Gaussian widths $\overrightarrow{\lambda}$ satisfy

$$m\ddot\lambda_i = -m\omega_i^2\lambda_i + \frac{\hbar^2}{m\lambda_i^3} - \frac{2}{\lambda_i}\langle r_i U_{r_i}^{\mathrm{mf}}\rangle. \tag{13}$$

The contact potential contributes to the rhs with $\langle r_i U_{r_i}^c\rangle = -g_c N/(4\sqrt{2}\pi^{3/2}\lambda_1\lambda_2\lambda_3)$, by yielding a gradient in Eq. (13). By setting $g_d = 0$ and $\lambda_i = R_i^G b_i$ with adequate constants R_i^G, Eq. (13) coincide with the previous differential system for the parameters b_i up to a term that vanishes in the Thomas-Fermi limit. The Gaussian ansatz, successful in both the dilute and the strongly interacting limits, is, thus, a valid interpolation between these regimes. This also shows the resilience of surface modes toward the sample profile. The dipolar potential also contributes through a gradient, thereby turning Eq. (13) into the equation of motion of a fictive pointlike particle of mass m, of position $\overrightarrow{\lambda}$, and experiencing an effective potential V^e:

$$V^e(\overrightarrow{\lambda}) = \sum_{i=1}^{3}\left(\frac{\hbar^2}{2m\lambda_i^2}+\frac{m}{2}\omega_i^2\lambda_i^2\right)+\frac{N}{\pi^{3/2}\lambda_1\lambda_2\lambda_3} \times\left[\frac{g_c}{2\sqrt{2}}+4\sqrt{2}g_d\int d\mathbf{r}\frac{1-3\cos^2\theta_{\mathbf{r}}}{r^2}\prod_{j=1}^{3}e^{-r_i^2/(2\lambda_i^2)}\right]. \tag{14}$$

This result perfectly matches the predictions of the variational methods [19,26], in which the same differential system $m\ddot{\lambda_i} = -dV^e(\overrightarrow{\lambda})/d\lambda_i$ was obtained. One retrieves the low-energy monopolar and quadrupolar frequencies for a cylindrical condensate obtained by other theoretical methods [19,27] and successfully compared with the experiments [17]. The method of moments combined with the TWE is, thus, entirely equivalent to the variational method for this class of Gaussian ansatz, but it nicely avoids the usual algebraic operations required in the latter to uncouple the Euler-Lagrange equations.

In conclusion, the classical phase-space evolution given by the TWE is sufficient to explain many important features in the dynamics of zero-temperature condensates. By using an adequate transformation of the Wigner distribution, we have recovered the dynamics of an expanding BEC in the Thomas-Fermi regime. With the method of moments, we have obtained the correct low-energy spectrum of a condensate in the presence of contact and dipolar interactions of arbitrary strength. Additionally, the universal oscillation frequency in the presence of a $1/r^2$ potential is also predicted by the TWE: This suggests that the hidden symmetry of the full equation [24] is preserved despite the truncation of the Wigner equation. This phase-space approach has allowed us to revisit the $ABCD$ law for the propagation of dilute and partially coherent matter waves, which can be extended to account for mean-field interactions [9]. Last, we have shown that the quantum terms of the Wigner equation do not affect the low-energy modes of trapped interacting condensates. Since quantum signatures appear in regimes where the TWE fails, the presented results suggest that neither the Thomas-Fermi nor the diluted limit are appropriate to evidence such signatures: Rather, these should be tracked with intermediate mean-field interactions or in high-order modes. Recently, it has been brought to our attention that the variational and moment methods have also been investigated in Ref. [28].

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